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이학박사 학위논문

Global regularity for
quasilinear elliptic equations
with Morrey data

(모리 데이터를 갖는 비선형 타원방정식의 대역적
정칙성)

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지도교수 변 순 식

이 논문을 이학박사 학위논문으로 제출함

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서울대학교 대학원

수리과학부

신필수

신필수의 이학박사 학위논문을 인준함

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위 원 장 _____ (인)

부 위 원 장 _____ (인)

위 원 _____ (인)

위 원 _____ (인)

위 원 _____ (인)

Global regularity for quasilinear elliptic equations with Morrey data

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
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by

Pilsoo Shin

Dissertation Director : Professor Sun-Sig Byun

Department of Mathematical Sciences
Seoul National University

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Abstract

In this thesis, we deal with general quasilinear m -Laplacian type operators in divergence form. The nonlinear terms are given by Carathéodory functions which are controlled within data belonging to suitable Morrey spaces.

We first prove global boundedness and Hölder continuity up to the boundary for the weak solutions of such equations, generalizing this way the classical L^p -result of Ladyzhenskaya and Ural'tseva to the settings of the Morrey spaces. The boundary of the underlying domain is supposed to satisfy a capacity density condition.

We also derive global gradient estimates in Morrey spaces for the weak solutions to quasilinear equations having (δ, R) -vanishing nonlinearity. In this case, we assume the boundary of the domain considered is Reifenberg flat which includes boundaries with rough fractal structure.

Key words: Quasilinear elliptic equation, Global Hölder continuity, Global gradient estimates, Morrey spaces, Irregular domain.

Student Number: 2011-30095

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Chapter 1

Introduction

In this thesis, we study general quasilinear elliptic equations with discontinuous ingredients whose prototype is the m -Laplace operator $\operatorname{div}(|Du|^{m-2}Du)$ with $m > 1$, and which are controlled within the Morrey functional scales. The main purpose of this thesis is to get sufficient conditions ensuring boundedness and Hölder continuity and to derive global gradient estimate in Morrey spaces up to the boundary for the weak solutions.

In Chapter 3, we deal with weak solutions $u \in W_0^{1,m}(\Omega)$ of the Dirichlet problem

$$\begin{cases} \operatorname{div}(\mathbf{a}(x, u, Du)) = b(x, u, Du) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with generally non-smooth boundary, $m \in (1, n]$, and $\mathbf{a}: \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $b: \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ are Carathéodory maps.

Regarding the nonlinear terms in (1.1), we assume *controlled growths* with respect to u and Du , that is,

$$\begin{cases} |\mathbf{a}(x, u, Du)| = \mathcal{O}\left(\varphi(x) + |u|^{\frac{\ell(m-1)}{m}} + |Du|^{m-1}\right), \\ |b(x, u, Du)| = \mathcal{O}\left(\psi(x) + |u|^{\ell-1} + |Du|^{\frac{m(\ell-1)}{\ell}}\right) \end{cases}$$

as $|u|, |Du| \rightarrow \infty$, where ℓ is the Sobolev conjugate of m , and *coercivity* of the differential operator considered

$$\mathbf{a}(x, u, Du) \cdot Du \geq \gamma|Du|^m - \Lambda|u|^\ell - \Lambda\varphi(x)^{\frac{m}{m-1}}$$

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with non-negative functions φ and ψ and constants $\gamma > 0$ and $\Lambda \geq 0$. It is worth noting that $\varphi \in L^{\frac{m}{m-1}}(\Omega)$ and $\psi \in L^{\frac{nm}{nm+m-n}}(\Omega)$, together with the controlled growths, are the *minimal* hypotheses on the data under which the concept of $W_0^{1,m}(\Omega)$ -weak solution to (1.1) makes sense. In what follows, we will assume that φ and ψ are non-negative measurable functions belonging to suitable *Morrey spaces*. Namely, we suppose

$$\begin{aligned} \varphi &\in L^{p,\lambda}(\Omega) \quad \text{with } p > \frac{m}{m-1}, \quad \lambda \in (0, n) \text{ and } (m-1)p + \lambda > n \\ \psi &\in L^{q,\mu}(\Omega) \quad \text{with } q > \frac{mn}{mn+m-n}, \quad \mu \in (0, n) \text{ and } mq + \mu > n. \end{aligned} \quad (1.2)$$

The non-regular boundary of Ω will be assumed to satisfy a *density condition* expressed in terms of variational P -capacity for some $P \in (1, m)$ (see (2.1) below), which requires the complement $\mathbb{R}^n \setminus \Omega$ to be *uniformly P -thick*. This notion is a natural generalization of the *measure density condition*, known also as *(A)-condition* of Ladyzhenskaya and Ural'tseva (cf. [18, 19, 20]), which holds for instance when each point of $\partial\Omega$ supports the *exterior cone property*, excluding this way exterior spikes on $\partial\Omega$. In that sense, the uniform P -thickness condition is satisfied by domains with C^1 -smooth or Lipschitz continuous boundaries, but it holds also when $\partial\Omega$ is flat in the sense of Reifenberg, including this way boundaries with fractal structure such as the von Koch snowflake. Anyway, the class of domains verifying the capacity density condition (2.1) goes beyond these common examples and contains for example sets with boundaries which support the *uniform corkscrew* condition.

In Chapter 4, we consider the Dirichlet problem

$$\begin{cases} \operatorname{div}(\mathbf{a}(x, u, Du) + \mathbf{b}(x, u)) = c(x, u, Du) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain, $m \in (1, n]$, and $\mathbf{a}: \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\mathbf{b}: \Omega \times \mathbb{R} \rightarrow \mathbb{R}^n$ and $c: \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ are Carathéodory maps.

Regarding the quasilinear operator in (1.3), we assume that it is *uniformly elliptic* with nonlinear terms satisfying the controlled growth conditions

$$\begin{cases} |\mathbf{b}(x, u)| = \mathcal{O}\left(\varphi(x) + |u|^{\frac{\ell(m-1)}{m}}\right), \\ |c(x, u, Du)| = \mathcal{O}\left(\psi(x) + |u|^{\ell-1} + |Du|^{\frac{m(\ell-1)}{\ell}}\right) \end{cases}$$

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as $|z|, |Du| \rightarrow \infty$, where non-negative functions φ and ψ are supposed in (1.2).

For what concerns the nonlinearity $\mathbf{a}(x, z, \xi)$, we require these to have (δ, R) -*vanishing* with respect to the spatial variable x and to be *local uniform continuous* with respect to z . These are natural generalization of the assumptions in [4, 30].

We assume that the non-regular boundary $\partial\Omega$ is *Reifenberg flat* which means that it is well approximated by hyperplanes at each point and at each scale (see Definition 2.2). This is a sort of “minimal regularity” of the boundary guaranteeing the main results of the geometric analysis continue to hold true in Ω . In particular, C^1 -smooth or Lipschitz continuous boundaries belong to that category, but the class of Reifenberg flat domains extends beyond these common examples and contains domains with rough fractal boundaries such as the Helge von Koch snowflake.

The regularity problem for solutions to (1.1) has been a long-standing problem in the PDEs theory, related to the Hilbert 19th Problem. In particular, the task to get Hölder continuity of the weak solutions under very general hypotheses on the data is a first step towards developing relevant solvability and regularity theory for (1.1) in the framework of various functional scales (see for instance [4, 5, 30] and the references therein). In case when (1.1) is the Euler–Lagrange equation of a given functional \mathcal{F} that is the problem of regularity of the minimizers of \mathcal{F} and this links (1.1) to important equations from differential geometry or mathematical physics, such as Gunzburg–Landau, nonlinear Schrödinger, non-Newtonian fluids and so on.

The Hilbert 19th Problem has been brilliantly solved by De Giorgi in [11] for $W_0^{1,2}$ -weak solutions to *linear* differential operators over Lipschitz continuous domains when $m = 2$, $\varphi \in L^p$ with $p > n$ and $\psi \in L^q$ with $2q > n$, and this provided the initial breakthrough in the modern theory of quasilinear equations in more than two independent variables. The De Giorgi result was extended to *linear* equations in the *non- L^p* settings (i.e., when a sort of (1.2) holds) by Morrey in [26] and Lewy and Stampacchia in [23] to equations with measures at the right-hand side, assuming $\varphi \in L^{2,\lambda}$, $\psi \in L^{1,\mu}$ with $\lambda, \mu > n - 2$. Moving to the quasilinear equation (1.1), we dispose of the seminal L^p -result of Serrin [35], which provides *interior* boundedness and Hölder continuity of the $W_0^{1,m}$ -weak solutions to (1.1) in the *sub-controlled* case when the nonlinearities grow as $|u|^{m-1} + |Du|^{m-1}$, and the behaviour with respect to x of $\mathbf{a}(x, u, Du)$ and $b(x, u, Du)$ is controlled in terms of φ

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and ψ , respectively, with

$$\begin{aligned} \varphi &\in L^p(\Omega) \quad \text{with } p > \frac{m}{m-1}, & (m-1)p > n \\ \psi &\in L^q(\Omega) \quad \text{with } q > \frac{mn}{mn+m-n}, & mq > n. \end{aligned} \tag{1.4}$$

Global boundedness of the $W_0^{1,m}$ -weak solutions to (1.1) with general nonlinearities of *controlled* growths has been obtained by Ladyzhenskaya and Ural'tseva in [18] under the hypotheses (1.4) and for domains satisfying the *measure density (A)-condition*. Assuming *natural growths* of the data (that is, $\mathbf{a}(x, u, Du) = \mathcal{O}(\varphi(x) + |Du|^{m-1})$ and $b(x, u, Du) = \mathcal{O}(\psi(x) + |Du|^m)$) and (1.4), Ladyzhenskaya and Ural'tseva proved later in [19] Hölder continuity up to the boundary for the *bounded* weak solutions of (1.1), and Gariepy and Ziemer extended in [12] their result to domains with *P-thick complements*. It was Trudinger [37] the first to get *global* Hölder continuity of the *bounded* solutions in the *non- L^p* settings under the *natural structure hypotheses* of Ladyzhenskaya and Ural'tseva with $\varphi \in L^{n/(m-1),\varepsilon}$, $\psi \in L^{n/m,\varepsilon}$ for a small $\varepsilon > 0$, while Lieberman derived in [24] a very general result on *interior* Hölder continuity when φ and ψ are suitable measures. We refer the author also to the works by Rakotoson [32], Rakotoson and Ziemer [33] and Zamboni [38] for various *interior* regularity results regarding the problem (1.1).

Chapter 3 is a natural continuation of [3] where boundedness has been proved for (1.1) with Morrey data in the case $m = 2$ under the two-sided (A) condition on $\partial\Omega$. Here we derive *global boundedness* (Theorem 3.1.1) and *Hölder continuity up to the boundary* (Theorem 3.1.3) for *each* $W_0^{1,m}(\Omega)$ -weak solution of the *coercive* Dirichlet problem (1.1) over domains with *P-thick* complements assuming *controlled growths* of the nonlinearities and Morrey data φ and ψ satisfying (1.2). Apart from the more general class of domains considered, we extend this way the classical *L^p -results* of Ladyzhenskaya and Ural'tseva [18, 19, 20] to the *non- L^p -settings* by weakening the hypotheses on φ and ψ to the scales of *Morrey type*. A comparison between (1.2) and (1.4) shows that the decrease of the degrees p and q of Lebesgue integrability of the data φ and ψ is at the expense of increase of the Morrey exponents λ and μ , and the range of these variations is always controlled by the relations $(m-1)p + \lambda > n$ and $mq + \mu > n$. Indeed, in the particular case $\lambda = \mu = 0$ and domains with exterior cone property, our results reduce to these of Ladyzhenskaya and Ural'tseva [18, 19, 20]. However, our Theorems 3.1.1 and 3.1.3 generalize substantially the results in [18, 19, 20] because even if

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$(m-1)p \leq n$ and $mq \leq n$, there exist functions $\varphi \in L^{p,\lambda}$ with $(m-1)p+\lambda > n$ and $\psi \in L^{q,\mu}$ with $mq+\mu > n$ for which (1.2) hold, but $\varphi \notin L^{p'} \forall p' > n/(m-1)$ and $\psi \notin L^{q'} \forall q' > n/m$ and therefore (1.4) fail. Moreover, as will be seen in Section 3.3 below, the *controlled* growths and the restrictions (1.2) on the Sobolev–Morrey exponents are *optimal* for the global boundedness and the subsequent Hölder continuity of the weak solutions to (1.1).

Global Calderón–Zygmund estimates for the $W_0^{1,2}$ -weak solutions to *linear* differential operators has been obtained by Byun and Wang in [7] with L^p settings in rough domains. This result was extended to *quasilinear* equations having *linear*-growth in [8] with L^p settings and Mengesha and Phuc in [25] under *weighted- L^p* settings. To the *nonlinear* equations of m -Laplacian type, Byun and Ryu proved it in [6] for *weighted- L^p* functions. *Global gradient estimates* for the $W_0^{1,2}$ -weak solutions to (1.3) with general nonlinearities of *subcontrolled* growths has been obtained by Palagachev and Softova in [30] for *Reifenberg* domains and Byun and Palagachev extended in [4] their result to *non- L^p* settings under *controlled structure hypotheses*.

The thesis is organized as follows. In Chapter 2, we start with introducing the concept of P -thickness and discuss its relations to the measure density property of $\partial\Omega$. We collect various auxiliary results which form the analytic heart of our approach. In Chapter 3, we prove *global boundedness* and *Hölder continuity*. We list in a detailed way the hypotheses imposed on the data of (1.1) and state the main results of the chapter. Of particular interest here is the Gehring–Giaquinta–Modica type Lemma 3.2.1 that asserts *better integrability* for the gradient of the weak solution over domains with P -thick complements, a particular case of which is due to Kilpeläinen and Koskela [17]. The proof of the *global boundedness* result (Theorem 3.1.1) is given in Section 3.3. Our technique relies on the De Giorgi approach to the boundedness as adapted by Ladyzhenskaya and Ural'tseva (cf. [20, Chapter IV]) to quasilinear equations. Namely, using the controlled growth assumptions, we get exact decay estimates for the total mass of the weak solution taken over its level sets. However, unlike the L^p -approach of Ladyzhenskaya and Ural'tseva, the mass we have to do with is taken with respect to a positive Radon measure \mathcal{M} , which depends not only on the Lebesgue measure, but also on $\varphi^{\frac{m}{m-1}}$, ψ and a suitable power of the weak solution itself. Thanks to the hypotheses (1.2), the measure \mathcal{M} allows to employ very precise inequalities of trace type due to D.R. Adams [1] and these lead to a bound of the \mathcal{M} -mass of u in terms of the m -energy of u . At this point we combine the controlled growth conditions with the better integrability of the gradient

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in order to estimate the m -energy of u in terms of small multiplier of the same quantity plus a suitable power of the level set \mathcal{M} -measure. The global boundedness of the weak solution then follows by a classical result known as *Hartman–Stampacchia maximum principle*. At the end of Section 3.3 we show sharpness of the controlled growths hypotheses as well as of (1.2) on the level of explicit examples built on quasilinear operators with m -Laplacean principal part. Section 3.4 is devoted to the proof of the *global Hölder continuity* as claimed in Theorem 3.1.3. Indeed, the boundedness of the weak solution is guaranteed by Theorem 3.1.1 and the fine results obtained by Lieberman in [24] apply to infer *interior Hölder continuity*. To extend it up to the boundary of Ω , we adopt to our situation the approach of Gariepy and Ziemer from [12] which relies on the Moser iteration technique in obtaining growth estimates for the gradient of the solution. The crucial step here is ensured by Lemma 3.4.1 which combines with the P -thickness condition in order to get estimate for the oscillation of u over small balls centered on $\partial\Omega$ in terms of a suitable positive power of the radius. Just for the sake of simplicity, we proved Theorem 3.1.3 under the controlled growths hypotheses. Following the same arguments, it is easy to see that the *global Hölder continuity* result still holds true for the *bounded* weak solutions of (1.1) if one assumes the *natural structure* conditions of Ladyzhenskaya and Ural'tseva instead of the controlled ones (cf. Theorem 3.4.2). In Chapter 4, we derive *global gradient estimates* in Morrey spaces. We, first of all, list the hypotheses imposed on the data of (1.3) and state the main result of the chapter. The technique employed in the proof of Theorem 4.1.1 relies on the weighted L^p -theory of non-linear divergence form elliptic equations with (δ, R) -vanishing nonlinearity (cf. [6]). We observe the modeled weighted L^p -theory in Section 4.2 and prove (δ, R) -vanishing properties of superposition operators in Section 4.3. The proof of Theorem 4.1.1 is given in Section 4.4. Adopting the approach of Byun and Palagachev from [4], we rewrite (1.3) into the modeled equation in Section 4.2. The better Morrey regularity ensured by Lemma 4.2.2 and the bootstrap argument gives the results of Theorem 4.1.1.

Chapter 2

Auxiliary Results

2.1 Notations

Throughout the thesis, we will use standard notations and will assume that the functions and sets considered are measurable.

We denote by $B_\rho(x)$ (or simply B_ρ if there is no ambiguity) the n -dimensional open ball with center $x \in \mathbb{R}^n$ and radius ρ . The Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ will be denoted by $|E|$ while, for any integrable function u defined on a set A , its integral average is given by

$$\bar{u}_A := \oint_A u(x) \, dx = \frac{1}{|A|} \int_A u(x) \, dx.$$

We will denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions over a bounded domain $\Omega \subset \mathbb{R}^n$ with compact support contained in that domain, and $L^p(\Omega)$ stands for the standard Lebesgue space with a given $p \in [1, \infty]$. The Sobolev space $W_0^{1,p}(\Omega)$ is defined, as usual, by the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)}$$

for $p \in [1, \infty)$.

Given $s \in [1, \infty)$ and $\theta \in [0, n]$, the Morrey space $L^{s,\theta}(\Omega)$ is the collection

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of all functions $u \in L^s(\Omega)$ such that

$$\|u\|_{L^{s,\theta}(\Omega)} := \sup_{x_0 \in \Omega, \rho > 0} \left(\rho^{-\theta} \int_{B_\rho(x_0) \cap \Omega} |u(x)|^s dx \right)^{1/s} < \infty.$$

The space $L^{s,\theta}(\Omega)$, equipped with the norm $\|\cdot\|_{L^{s,\theta}(\Omega)}$ is Banach space and the limit cases $\theta = 0$ and $\theta = n$ give rise, respectively, to $L^s(\Omega)$ and $L^\infty(\Omega)$.

For $0 < \alpha < n$, the Riesz potential $I_\alpha f$ of a locally integrable function f on \mathbb{R}^n is defined by

$$I_\alpha f(x) := \frac{1}{c_\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

where the constant c_α is given by

$$c_\alpha = \pi^{n/2} 2^\alpha \frac{\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $n \geq 2$. In order to set down the requirements on $\partial\Omega$, we need to recall the concept of *variational p -capacity* of a set for $1 < p < \infty$. Thus, given a compact set $C \subset \Omega$, its p -capacity is defined as

$$\text{Cap}_p(C, \Omega) = \inf_g \int_{\Omega} |Dg|^p dx$$

where the infimum is taken over all functions $g \in C_0^\infty(\Omega)$ such that $g = 1$ in C . If $U \subset \Omega$ is an open set, then

$$\text{Cap}_p(U, \Omega) = \sup_{C \subset U} \text{Cap}_p(C, \Omega), \quad C \text{ is compact},$$

while, if $E \subset \Omega$ is an arbitrary set, then

$$\text{Cap}_p(E, \Omega) = \inf_{E \subset U \subset \Omega} \text{Cap}_p(U, \Omega), \quad U \text{ is open}.$$

In particular, if $E \subset E' \subset \Omega'$ then

$$\text{Cap}_p(E, \Omega) \leq \text{Cap}_p(E', \Omega')$$

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and, in case of two concentric balls B_R and B_r with $R > r$, the next formula

$$\text{Cap}_p(\overline{B}_r, B_R) = Cr^{n-p}$$

is known for $p > 1$, where $C > 0$ depends on n, p and R/r (see [15, Chapter 2] for more details).

In the sequel we will suppose that the complement $\mathbb{R}^n \setminus \Omega$ of Ω satisfies the next *uniform P -thickness condition* for some $P \in (1, m)$: *there exist positive constants A_Ω and r_0 such that*

$$\text{Cap}_P((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_r(x), B_{2r}(x)) \geq A_\Omega \text{Cap}_P(\overline{B}_r(x), B_{2r}(x)) \quad (2.1)$$

for all $x \in \mathbb{R}^n \setminus \Omega$ and all $r \in (0, r_0)$.

Let us point out that replacing the capacity above with the Lebesgue measure, (2.1) reduces to the measure density condition (the *(A)-condition* of Ladyzhenskaya and Ural'tseva) which holds for instance when Ω supports the *uniform exterior cone property*. If a given set E satisfies the measure density condition then it is uniformly p -thick for each $p > 1$, whereas each nonempty set is uniformly p -thick if $p > n$. Further on, a uniformly q -thick set is also uniformly p -thick for all $p \geq q$ and, as proved in [22], the uniformly p -thick sets have a deep self-improving property to be uniformly q -thick for *some* $q < p$, depending on n, p and the constant of the p -thickness. In this sense, it is not restrictive to ask $P < m$ in (2.1) since even if $\mathbb{R}^n \setminus \Omega$ were m -thick, the existence of a $P < m$ verifying (2.1) is ensured by [22]. Yet another example of uniformly p -thick sets for all $p > 1$ is given by those satisfying the uniform *corkscrew* condition: a set E is uniformly corkscrew if there exist constants $C > 0$ and $r_0 > 0$ such that for any $x \in E$ and any $r \in (0, r_0)$ there is a point $y \in B_r(x) \setminus E$ with the property that $B_{r/C}(y) \subset \mathbb{R}^n \setminus E$.

We will also consider a bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, the boundary $\partial\Omega$ of which is *Reifenberg flat* in the sense of the following definition.

Definition 2.1.1. *The domain Ω is said to be (δ, R) -Reifenberg flat if there exist positive constants δ and R with the property that for each $x_0 \in \partial\Omega$ and each $\rho \in (0, R)$ there is a local coordinate system $\{x_1, \dots, x_n\}$ with origin at the point x_0 and such that*

$$B_\rho(x_0) \cap \{x : x_n > \rho\delta\} \subset B_\rho(x_0) \cap \Omega \subset B_\rho(x_0) \cap \{x : x_n > -\rho\delta\} \quad (2.2)$$

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Let us note that the above definition makes sense when $0 < \delta < 2^{-n-1}$ (see [36]).

It is clear that this definition is significant only for small values of δ while, by the scaling invariance property, R could be any constant greater than 1. The Reifenberg flatness means that the boundary $\partial\Omega$ is well approximated by hyperplanes at every point and at every scale and it was Reifenberg who arrived first to that concept in his studies [34] on Plateau problems, proving that such a domain is locally a topological disc when δ is small enough. Moreover, he showed that a subset of \mathbb{R}^n which is well approximated by m -dimensional hyperplanes at each point and at each scale is locally a bi-Hölder image of the unit ball in \mathbb{R}^m . These results have been extended recently by David et al. [10] to area-minimizing cones, proving that a subset of \mathbb{R}^3 which is well approximated in the sense of Hausdorff distance by one of the three standard area-minimizing cones at each point and at each small scale is locally a bi-Hölder deformation of a minimal cone.

The Reifenberg flatness exhibits a kind of “minimal regularity requirement” on the boundary $\partial\Omega$ which ensures the validity in Ω of the main natural properties of the geometric analysis (the reader is referred to the excellent survey by Toro [36]). Let us note that each domain with at least C^1 -smooth boundary is Reifenberg flat with δ vanishing when $R \rightarrow 0+$. Another important example is that of a domain Ω with boundary $\partial\Omega$ which is locally given as graph of a Lipschitz continuous function with small (*less than* δ) Lipschitz constant. Really, the class of Reifenberg flat domains is much wider and contains domains with rough fractal boundaries. For instance, the von Koch snowflake is a Reifenberg flat when the angle of the spike with respect to the horizontal is small enough (cf. [36] once again).

2.2 Basic tools

Proposition 2.2.1. (Embeddings between Morrey spaces, see [31]) *For arbitrary $s', s'' \in [1, \infty)$ and $\theta', \theta'' \in [0, n)$, one has*

$$L^{s', \theta'}(\Omega) \subseteq L^{s'', \theta''}(\Omega)$$

if and only if

$$s' \geq s'' \geq 1 \quad \text{and} \quad \frac{s'}{n - \theta'} \geq \frac{s''}{n - \theta''}.$$

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Proposition 2.2.2. (Hartman–Stampacchia maximum principle, see [16], [20]*Chapter II, Lemma 5.1) *Let $\tau: \mathbb{R} \rightarrow [0, \infty)$ be a non-increasing function and suppose there exist constants $C > 0$, $k_0 \geq 0$, $\delta > 0$ and $\alpha \in [0, 1 + \delta]$ such that*

$$\int_k^\infty \tau(t) dt \leq C k^\alpha (\tau(k))^{1+\delta} \quad \forall k \geq k_0.$$

Then τ supports the finite time extinction property, that is, there is a number k_{\max} , depending on C , k_0 , δ , α and $\int_{k_0}^\infty \tau(t) dt$, such that

$$\tau(k) = 0 \quad \forall k \geq k_{\max}.$$

Proposition 2.2.3. (Adams trace inequality, see [1]) *Let \mathcal{M} be a positive Radon measure supported in Ω and such that $\mathcal{M}(B_\rho(x)) \leq K \rho^{\alpha_0}$ for each $x \in \mathbb{R}^n$ and each $\rho > 0$, where K is an absolute constant and*

$$\alpha_0 = \frac{s}{r}(n - r), \quad 1 < r < s < \infty, \quad r < n.$$

Then

$$\left(\int_\Omega |v(x)|^s d\mathcal{M} \right)^{1/s} \leq C(n, s, r) K^{1/s} \left(\int_\Omega |Dv(x)|^r dx \right)^{1/r} \quad \forall v \in W_0^{1,r}(\Omega).$$

In particular, if $d\mathcal{M} = c(x) dx$ with $c \in L^{1, n-r+\varepsilon_0}(\Omega)$ and $\varepsilon_0 > 0$, then

$$\left(\int_\Omega |v(x)|^s c(x) dx \right)^{1/s} \leq C(n, s, r, \|c\|_{L^{1, n-r+\varepsilon_0}(\Omega)}) \left(\int_\Omega |Dv(x)|^r dx \right)^{1/r}$$

for all $v \in W_0^{1,r}(\Omega)$, where $n - r + \varepsilon_0 = \frac{s}{r}(n - r)$, $1 < r < s < \infty$, $r < n$.

Proposition 2.2.4. (Gehring–Giaquinta–Modica lemma, see [13, Proposition 1.1, Chapter V]) *Let B be a fixed ball and $G \in L^s(B)$, $F \in L^{s_0}(B)$ be nonnegative functions with $s_0 > s > 1$. Suppose*

$$\int_{B_\rho} G^s(x) dx \leq c \left(\int_{B_{2\rho}} G(x) dx \right)^s + \int_{B_{2\rho}} F^s(x) dx + \theta \int_{B_{2\rho}} G^s(x) dx$$

for each ball B_ρ of radius $\rho \in (0, \rho_0)$ such that $B_{2\rho} \subset B$, where $0 \leq \theta < 1$.

Then there exist constants C and $m_0 \in (s, s_0]$, depending on n , c , s , s_0

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and θ , such that

$$\left(\int_{B_\rho} G^{m_0}(x) dx \right)^{1/m_0} \leq C \left(\left(\int_{B_{2\rho}} G^s(x) dx \right)^{1/s} + \left(\int_{B_{2\rho}} F^{s_0}(x) dx \right)^{1/s_0} \right).$$

Proposition 2.2.5. (John–Nirenberg lemma, see [37, Lemma 1.2], [14, Theorem 7.21]) *Let B_0 be a ball in \mathbb{R}^n , $u \in W^{1,m}(B_0)$ and suppose that, for any ball $B \subset B_0$ with the same center as B_0 there exists a constant K such that*

$$\|Du\|_{L^m(B)} \leq K|B|^{\frac{n-m}{mn}}.$$

Then there exists constants $\sigma_0 > 0$ and C depending on K, m, n such that

$$\int_{B_0} e^{\sigma_0 u} dx \int_{B_0} e^{-\sigma_0 u} dx \leq C|B_0|^2.$$

Proposition 2.2.6. (see [14, Lemma 8.23]) *Let F and G be nondecreasing functions in an interval $(0, R]$. Suppose that for all $\rho \leq R$ one has*

$$G(\rho/2) \leq c_0(G(\rho) + F(\rho))$$

for some $0 < c_0 < 1$. Then for any $0 < \tau < 1$ and $\rho \leq R$ we have

$$G(\rho) \leq C \left(\left(\frac{\rho}{R} \right)^\alpha G(R) + F(\rho^\tau R^{1-\tau}) \right)$$

where $C = C(c_0)$ and $\alpha = \alpha(c_0, \tau)$ are positive constants.

Proposition 2.2.7. (see [2]) *Let $s \in (1, \infty)$, $\theta \in [0, n)$ and $f \in L^{s,\theta}(\Omega)$. Then the Riesz potential $I_1 f$ of f is also contained in Morrey space $L^{s_\theta^*, \theta}(\Omega)$ and*

$$\|I_1 f\|_{L^{s_\theta^*, \theta}(\Omega)} \leq C \|f\|_{L^{s,\theta}(\Omega)}.$$

Proposition 2.2.8. (Regularity of functions with gradients in Morrey spaces, see [9, Lemma 3.III, Lemma 3.IV]) *Assume that Ω is Reifenberg flat domain and let $u \in W^{1,s}$ with $Du \in L^{s,\theta}$, $\theta \in [0, n)$. Then the following assertions hold true: (1) If $s + \theta < n$ then $u \in L^{\frac{ns}{n-s}, \frac{n\theta}{n-s}}(\Omega) \subset L^{s,\theta+s}(\Omega)$ with*

$$\|u\|_{L^{s,\theta+s}(\Omega)} \leq \|u\|_{L^{\frac{ns}{n-s}, \frac{n\theta}{n-s}}(\Omega)} \leq C(n, s, \theta) \left(\|u\|_{L^s(\Omega)} + \|Du\|_{L^{s,\theta}(\Omega)} \right).$$

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(2) If $s + \theta = n$ then $u \in L^{s', \theta'}(\Omega)$ for any $s' < \infty$ and any $\theta' < n$, and

$$\|u\|_{L^{s', \theta'}(\Omega)} \leq C(n, s, \theta) \left(\|u\|_{L^s(\Omega)} + \|Du\|_{L^{s, \theta}(\Omega)} \right).$$

(3) If $s + \theta > n$ then $u \in C^{0, \alpha}(\overline{\Omega})$ with $\alpha = 1 - \frac{n-\theta}{s}$ and

$$\sup_{\overline{\Omega}} |u(x)| + \sup_{x, y \in \overline{\Omega}, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C(n, s, \theta) \left(\|u\|_{L^s(\Omega)} + \|Du\|_{L^{s, \theta}(\Omega)} \right).$$

2.3 Boundary Sobolev inequality

The next result is a boundary variant of the Sobolev inequality which holds under the P -thickness condition.

Lemma 2.3.1. (Boundary Sobolev inequality) *Let Ω be a bounded domain with uniformly P -thick complement $\mathbb{R}^n \setminus \Omega$ and consider a function $u \in W_0^{1, m}(\Omega)$ which is extended as zero outside Ω .*

Let B_ρ be a ball of radius $\rho \in (0, r_0/(1 - \theta))$, centered at a point of Ω and suppose $B_{\theta\rho} \setminus \Omega \neq \emptyset$ for some $0 < \theta < 1$.

Then for any $s \in [P, m]$ there is a constant $C = C(n, \theta, s, P, A_\Omega)$ such that

$$\left(\int_{B_\rho} |u(x)|^{\tilde{s}} dx \right)^{1/\tilde{s}} \leq C\rho \left(\int_{B_\rho} |Du(x)|^s dx \right)^{1/s} \quad (2.3)$$

for each $\tilde{s} \in [s, s^*]$, where s^* is the Sobolev conjugate of s ($s^* = ns/(n - s)$ if $s < n$ and s^* is any exponent greater than n otherwise).

Proof. Without loss of generality we may suppose that u is an s -quasicontinuous function in $W^{1, s}(B_\rho)$. Since $B_{\theta\rho} \setminus \Omega \neq \emptyset$, we can take a ball $B_{(1-\theta)\rho}(x_0)$ of radius $(1 - \theta)\rho$, centered at $x_0 \in \partial\Omega$ and such that $B_{(1-\theta)\rho}(x_0) \subset B_\rho$. Setting $\mathcal{N}(u) = \{x \in B_\rho : u(x) = 0\}$ and applying the Hölder inequality and [17, Lemma 3.1], we get

$$\begin{aligned} \left(\int_{B_\rho} |u(x)|^{\tilde{s}} dx \right)^{1/\tilde{s}} &\leq \left(\int_{B_\rho} |u(x)|^{s^*} dx \right)^{1/s^*} \\ &\leq C \left(\frac{1}{\text{Cap}_s(\mathcal{N}(u), B_{2\rho})} \int_{B_\rho} |Du(x)|^s dx \right)^{1/s} \end{aligned} \quad (2.4)$$

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whenever $s < n$. Indeed, (2.4) holds also for any $s^* > n$ when $s = n$. In fact,

$$\text{Cap}_{s'}(\mathcal{N}(u), B_{2\rho}) \leq C\rho^{n-s'} \text{Cap}_n(\mathcal{N}(u), B_{2\rho})^{s'/n}$$

for arbitrary $s' < n$, whence we have

$$\left(\frac{1}{\text{Cap}_{s'}(\mathcal{N}(u), B_{2\rho})} \int_{B_\rho} |Du|^{s'} dx \right)^{1/s'} \leq \left(\frac{1}{\text{Cap}_n(\mathcal{N}(u), B_{2\rho})} \int_{B_\rho} |Du|^n dx \right)^{1/n}.$$

Taking $s' = \frac{ns^*}{n+s^*} < n$ in (2.4) and using the above inequality, we get (2.4) for $s = n$ and for arbitrary $s^* > n$.

Since $u = 0$ in $\mathbb{R}^n \setminus \Omega$ except of a set of s -capacity zero and $B_{(1-\theta)\rho}(x_0) \subset B_\rho$, we have

$$\text{Cap}_s(\mathcal{N}(u), B_{2\rho}) \geq \text{Cap}_s(B_\rho \setminus \Omega, B_{2\rho}) \geq \text{Cap}_s(B_{(1-\theta)\rho}(x_0) \setminus \Omega, B_{2\rho})$$

by the properties of capacity, whereas

$$\text{Cap}_s(B_{(1-\theta)\rho}(x_0) \setminus \Omega, B_{2\rho}) \geq C(n, \theta, s) \text{Cap}_s(B_{(1-\theta)\rho}(x_0) \setminus \Omega, B_{2(1-\theta)\rho}(x_0)). \quad (2.5)$$

In fact, to see (2.5) we take functions $v \in C_0^\infty(B_{2\rho})$, $0 \leq v \leq 1$, $v = 1$ on $B_{(1-\theta)\rho}(x_0) \setminus \Omega$ and $\eta \in C_0^\infty(B_{2(1-\theta)\rho}(x_0))$, $|D\eta| \leq \frac{c}{(1-\theta)\rho}$, $0 \leq \eta \leq 1$, $\eta = 1$ on $B_{(1-\theta)\rho}(x_0)$. Then $v\eta \in C_0^\infty(B_{2(1-\theta)\rho}(x_0))$, $0 \leq v\eta \leq 1$, $v\eta = 1$ on $B_{(1-\theta)\rho}(x_0) \setminus \Omega$ and therefore, if $s < n$, we have

$$\begin{aligned} & \text{Cap}_s(B_{(1-\theta)\rho}(x_0) \setminus \Omega, B_{2(1-\theta)\rho}(x_0)) \\ & \leq \int_{B_{2(1-\theta)\rho}(x_0)} |D(v\eta)|^s dx \\ & \leq C \left(\int_{B_{2(1-\theta)\rho}(x_0)} |Dv|^s dx + \left(\frac{1}{(1-\theta)\rho} \right)^s \int_{B_{2(1-\theta)\rho}(x_0)} |v|^s dx \right) \\ & \leq C \left(\int_{B_{2\rho}} |Dv|^s dx + \left(\int_{B_{2\rho}} |v|^{s^*} dx \right)^{s/s^*} \right) \\ & \leq C \int_{B_{2\rho}} |Dv|^s dx. \end{aligned} \quad (2.6)$$

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The same bound holds true also if $s = n$ with a constant C , depending on θ in addition. Actually, making use of the Hölder and Sobolev ([15, 15.30]) inequalities, for arbitrary $t > 1$ we get

$$\begin{aligned}
\left(\frac{1}{(1-\theta)\rho}\right)^n \int_{B_{2(1-\theta)\rho}(x_0)} |v|^n dx &\leq \left(\frac{1}{(1-\theta)\rho}\right)^n \left(\int_{B_{2\rho}} |v|^{nt} dx\right)^{1/t} |B_{2\rho}|^{1-1/t} \\
&\leq \frac{1}{(1-\theta)^n} \rho^{-\frac{n}{t}} \left(\int_{B_{2\rho}} |v|^{nt} dx\right)^{1/t} = \frac{1}{(1-\theta)^n} \left(\int_{B_{2\rho}} |v|^{nt} dx\right)^{1/t} \\
&\leq \frac{C}{(1-\theta)^n} \rho^n \int_{B_{2\rho}} |Dv|^n dx = \frac{C}{(1-\theta)^n} \int_{B_{2\rho}} |Dv|^n dx
\end{aligned}$$

and thus (2.6) with $s = n$.

This way, (2.5) follows after taking the infimum in the right-hand side of (2.6) over all $v \in C_0^\infty(B_{2\rho})$ such that $v = 1$ in $B_{(1-\theta)\rho}(x_0) \setminus \Omega$.

Further on, the uniform s -thickness condition (2.1) yields

$$\begin{aligned}
&\text{Cap}_s(B_{(1-\theta)\rho}(x_0) \setminus \Omega, B_{2(1-\theta)\rho}(x_0)) \\
&\geq C(n, s, P, A_\Omega) \text{Cap}_s(B_{(1-\theta)\rho}(x_0), B_{2(1-\theta)\rho}(x_0)) \\
&= C(n, \theta, s, P, A_\Omega) \rho^{n-s}
\end{aligned}$$

and therefore the desired estimate (2.3) follows from (2.5) and (2.4). \square

Chapter 3

Boundedness and Hölder Continuity

3.1 Hypotheses and Main Results

In this chapter, we consider the Dirichlet problem

$$\begin{cases} \operatorname{div}(\mathbf{a}(x, u, Du)) = b(x, u, Du) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

given in (1.1). The nonlinearities considered are given by the Carathéodory maps $\mathbf{a}: \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $b: \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, where $\mathbf{a}(x, z, \xi) = (a^1(x, z, \xi), \dots, a^n(x, z, \xi))$. In other words, the functions $a^i(x, z, \xi)$ and $b(x, z, \xi)$ are measurable with respect to $x \in \Omega$ for all $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and are continuous with respect to $z \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ for almost all (a.a.) $x \in \Omega$. Moreover, we suppose:

- *Controlled growth conditions:* There exist a constant $\Lambda > 0$ and non-negative functions $\varphi \in L^{p,\lambda}(\Omega)$ with $p > \frac{m}{m-1}$, $\lambda \in (0, n)$ and $(m-1)p + \lambda > n$, and $\psi \in L^{q,\mu}(\Omega)$ with $q > \frac{mn}{mn+m-n}$, $\mu \in (0, n)$ and $mq + \mu > n$, such that

$$\begin{cases} |\mathbf{a}(x, z, \xi)| \leq \Lambda \left(\varphi(x) + |z|^{\frac{\ell(m-1)}{m}} + |\xi|^{m-1} \right), \\ |b(x, z, \xi)| \leq \Lambda \left(\psi(x) + |z|^{\ell-1} + |\xi|^{\frac{m(\ell-1)}{\ell}} \right) \end{cases} \quad (3.1)$$

for a.a. $x \in \Omega$ and all $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$. Here, ℓ is the Sobolev conjugate of m

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and is given by

$$\ell = \begin{cases} \frac{nm}{n-m} & \text{if } m < n, \\ \text{any exponent } \ell > n & \text{if } m = n. \end{cases} \quad (3.2)$$

- *Coercivity condition:* There exists a constant $\gamma > 0$ such that

$$\mathbf{a}(x, z, \xi) \cdot \xi \geq \gamma |\xi|^m - \Lambda |z|^\ell - \Lambda \varphi(x)^{\frac{m}{m-1}} \quad (3.3)$$

for a.a. $x \in \Omega$ and all $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

Recall that a function $u \in W_0^{1,m}(\Omega)$ is called *weak solution* to the Dirichlet problem (1.1) if

$$\int_{\Omega} \mathbf{a}(x, u(x), Du(x)) \cdot Dv(x) \, dx + \int_{\Omega} b(x, u(x), Du(x))v(x) \, dx = 0 \quad (3.4)$$

for each test function $v \in W_0^{1,m}(\Omega)$. It is worth noting that the convergence of the integrals involved in (3.4) for all admissible u and v is ensured by (3.1) under the *sole* assumptions $p \geq \frac{m}{m-1}$ and $q \geq \frac{mn}{mn+m-n}$ when $m < n$, $q > 1$ if $m = n$.

Throughout this chapter the omnibus phrase “*known quantities*” means that a given constant depends on the data in hypotheses (2.1)–(3.3), which include $n, m, \ell, \gamma, \Lambda, p, \lambda, q, \mu, \|\varphi\|_{L^{p,\lambda}(\Omega)}, \|\psi\|_{L^{q,\mu}(\Omega)}, P, \text{diam } \Omega, A_\Omega$ and r_0 . We will denote by C a generic constant, depending on known quantities, which may vary within the same formula.

Our first result claims *global essential boundedness* of the weak solutions to the problem (1.1).

Theorem 3.1.1. *Let Ω satisfy (2.1) and assume (3.1) and (3.3). Then each $W_0^{1,m}(\Omega)$ -weak solution to the problem (1.1) is globally essentially bounded. That is, there exists a constant M , depending on known quantities, on $\|Du\|_{L^m(\Omega)}$ and on the uniform integrability of $|Du|^m$, such that*

$$\|u\|_{L^\infty(\Omega)} \leq M. \quad (3.5)$$

An immediate consequence of Theorem 3.1.1 and the local properties of solutions to quasilinear elliptic equations (cf. [24, 38]) is the *interior Hölder continuity* of the weak solutions.

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Corollary 3.1.2. *Under the hypotheses of Theorem 3.1.1, each weak solution to (1.1) is locally Hölder continuous in Ω . That is,*

$$\sup_{x,y \in \Omega', x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq H \quad \forall \Omega' \Subset \Omega$$

with an exponent $\alpha \in (0, 1)$ and a constant $H > 0$ depending on the same quantities as M in (3.5) and on $\text{dist}(\Omega', \partial\Omega)$ in addition.

What really turns out is that assumptions (2.1), (3.1) and (3.3) are also sufficient to ensure Hölder continuity of the weak solutions *up to the boundary*, and this is the essence of our second main result.

Theorem 3.1.3. *Assume (2.1), (3.1) and (3.3). Then each weak solution of the Dirichlet problem (1.1) is globally Hölder continuous in Ω . Precisely,*

$$\sup_{x,y \in \bar{\Omega}, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq H,$$

where the exponent $\alpha \in (0, 1)$ and the Hölder constant $H > 0$ depend on the same quantities as M in (3.5).

3.2 Higher integrability of the gradient

The next result provides a crucial step to obtain global boundedness of the weak solutions to (1.1) although it is interesting by its own. Actually, it shows that the gradient of the weak solution to controlled growths and coercive problems (1.1) gains better integrability over domains with P -thick complements.

Lemma 3.2.1. *Assume (2.1), (3.1) and (3.3), and let $u \in W_0^{1,m}(\Omega)$ be a weak solution to the Dirichlet problem (1.1).*

Then there exist exponents $m_0 > m$ and $\ell_0 > \ell$ such that $u \in W^{1,m_0}(\Omega) \cap L^{\ell_0}(\Omega)$ and

$$\|Du\|_{L^{m_0}(\Omega)} + \|u\|_{L^{\ell_0}(\Omega)} \leq C \tag{3.6}$$

with a constant C depending on known quantities, on $\|Du\|_{L^m(\Omega)}$ and on the uniform integrability of $|Du|^m$ in Ω .

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Proof. Without loss of generality, we assume that the solution u and the data φ and ψ are extended as zero outside Ω . Let $x_0 \in \Omega$ be an arbitrary point and consider the concentric balls $B_\rho \subset B_{2\rho}$ centered at x_0 with $2\rho \in (0, r_0)$.

Case 1: $B_{3\rho/2} \setminus \Omega = \emptyset$. We have $B_{3\rho/2} \subset \Omega$ and let $\zeta \in C_0^\infty(B_{3\rho/2})$ be a cut-off function with the properties $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on B_ρ and $|D\zeta| \leq c/\rho$. Employing (3.4) with a test function $v(x) = \left(u(x) - \bar{u}_{B_{3\rho/2}}\right) \zeta^2(x)$, we get

$$\begin{aligned} \int_{\Omega} a^i(x, u(x), Du(x)) \left(\zeta^2(x) D_i u(x) + 2 \left(u(x) - \bar{u}_{B_{3\rho/2}} \right) \zeta(x) D_i \zeta(x) \right) dx \\ + \int_{\Omega} b(x, u(x), Du(x)) \left(u(x) - \bar{u}_{B_{3\rho/2}} \right) \zeta^2(x) dx = 0. \end{aligned}$$

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Thus (3.1), (3.3) and the choice of ζ lead to

$$\begin{aligned}
 \gamma \int_{B_\rho} |Du(x)|^m dx &\leq \Lambda \int_{B_{3\rho/2}} \varphi(x)^{\frac{m}{m-1}} dx + \underbrace{\Lambda \int_{B_{3\rho/2}} |u(x)|^\ell dx}_{I_1} \\
 &\quad + \underbrace{2\Lambda \int_{B_{3\rho/2}} \varphi(x) |u(x) - \bar{u}_{B_{3\rho/2}}| |D\zeta(x)| dx}_{I_2} \\
 &\quad + \underbrace{2\Lambda \int_{B_{3\rho/2}} |u(x)|^{\frac{\ell(m-1)}{m}} |u(x) - \bar{u}_{B_{3\rho/2}}| |D\zeta(x)| dx}_{I_3} \\
 &\quad + \underbrace{2\Lambda \int_{B_{3\rho/2}} |Du(x)|^{m-1} |u(x) - \bar{u}_{B_{3\rho/2}}| |D\zeta(x)| dx}_{I_4} \\
 &\quad + \underbrace{\Lambda \int_{B_{3\rho/2}} \psi(x) |u(x) - \bar{u}_{B_{3\rho/2}}| dx}_{I_5} \\
 &\quad + \underbrace{\Lambda \int_{B_{3\rho/2}} |u(x)|^{\ell-1} |u(x) - \bar{u}_{B_{3\rho/2}}| dx}_{I_6} \\
 &\quad + \underbrace{\Lambda \int_{B_{3\rho/2}} |Du(x)|^{\frac{m(\ell-1)}{\ell}} |u(x) - \bar{u}_{B_{3\rho/2}}| dx}_{I_7}.
 \end{aligned} \tag{3.7}$$

It follows from the triangle inequality that

$$I_1 = \int_{B_{3\rho/2}} |u(x)|^\ell dx \leq C \int_{B_{3\rho/2}} |u(x) - \bar{u}_{B_{3\rho/2}}|^\ell dx + C \int_{B_{3\rho/2}} |\bar{u}_{B_{3\rho/2}}|^\ell dx.$$

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In view of the Sobolev–Poincaré inequality, we get

$$\begin{aligned} \int_{B_{3\rho/2}} |u(x) - \bar{u}_{B_{3\rho/2}}|^\ell dx &\leq C\rho^{\ell(n(\frac{1}{\ell}-\frac{1}{m})+1)} \left(\int_{B_{3\rho/2}} |Du(x)|^m dx \right)^{\ell/m} \\ &= C\rho^{\ell(n(\frac{1}{\ell}-\frac{1}{m})+1)} \left(\int_{B_{3\rho/2}} |Du(x)|^m dx \right)^{\ell/m-1} \int_{B_{3\rho/2}} |Du(x)|^m dx, \end{aligned}$$

while the Hölder inequality implies

$$\int_{B_{3\rho/2}} |\bar{u}_{B_{3\rho/2}}|^\ell dx = |B_{3\rho/2}| \left| \int_{B_{3\rho/2}} u(x) dx \right|^\ell \leq |B_{3\rho/2}| \left(\int_{B_{3\rho/2}} |u(x)|^{\frac{\widehat{m}}{m}} dx \right)^{m/\widehat{m}}$$

where $\widehat{m} := \max \left\{ \frac{nm}{n+m}, 1 \right\}$. Hence, the term I_1 is estimated as follows

$$\begin{aligned} I_1 &\leq C\rho^{\ell(n(\frac{1}{\ell}-\frac{1}{m})+1)} \left(\int_{B_{3\rho/2}} |Du(x)|^m dx \right)^{\ell/m-1} \int_{B_{3\rho/2}} |Du(x)|^m dx \\ &\quad + C|B_{3\rho/2}| \left(\int_{B_{3\rho/2}} |u(x)|^{\frac{\widehat{m}}{m}} dx \right)^{m/\widehat{m}}. \end{aligned}$$

Using the Young and the Sobolev–Poincaré inequalities, as well as $|D\zeta| \leq c/\rho$, we get the bound

$$\begin{aligned} I_2 &= \int_{B_{3\rho/2}} \varphi(x) |u(x) - \bar{u}_{B_{3\rho/2}}| |D\zeta(x)| dx \\ &\leq C \int_{B_{3\rho/2}} |u(x) - \bar{u}_{B_{3\rho/2}}|^m |D\zeta(x)|^m dx + C \int_{B_{3\rho/2}} \varphi(x)^{\frac{m}{m-1}} dx \\ &\leq \frac{C}{\rho^m} \int_{B_{3\rho/2}} |u(x) - \bar{u}_{B_{3\rho/2}}|^m dx + C \int_{B_{3\rho/2}} \varphi(x)^{\frac{m}{m-1}} dx \\ &\leq C|B_{3\rho/2}| \left(\int_{B_{3\rho/2}} |Du(x)|^{\widehat{m}} dx \right)^{m\widehat{m}} + C \int_{B_{3\rho/2}} \varphi(x)^{\frac{m}{m-1}} dx. \end{aligned}$$

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In a similar manner one has

$$\begin{aligned}
 I_3 &= \int_{B_{3\rho/2}} |u(x)|^{\frac{\ell(m-1)}{m}} |u(x) - \bar{u}_{B_{3\rho/2}}| |D\zeta(x)| \, dx \\
 &\leq C|B_{3\rho/2}| \left(\int_{B_{3\rho/2}} |Du(x)|^{\widehat{m}} \, dx \right)^{m/\widehat{m}} + C \int_{B_{3\rho/2}} |u(x)|^\ell \, dx \\
 &= C|B_{3\rho/2}| \left(\int_{B_{3\rho/2}} |Du(x)|^{\widehat{m}} \, dx \right)^{m/\widehat{m}} + CI_1
 \end{aligned}$$

and

$$\begin{aligned}
 I_4 &= \int_{B_{3\rho/2}} |Du(x)|^{m-1} |u(x) - \bar{u}_{B_{3\rho/2}}| |D\zeta(x)| \, dx \\
 &\leq C(\varepsilon)|B_{3\rho/2}| \left(\int_{B_{3\rho/2}} |Du(x)|^{\widehat{m}} \, dx \right)^{m/\widehat{m}} + \varepsilon \int_{B_{3\rho/2}} |Du(x)|^m \, dx
 \end{aligned}$$

with arbitrary $\varepsilon > 0$.

To go further, we take $t = \ell = \frac{nm}{n-m}$ if $m < n$ and any $t > \max\{\frac{q}{q-1}, m\}$ otherwise, and apply successively the Hölder, Sobolev–Poincaré and Young inequalities. Thus, the following bound

$$\begin{aligned}
 I_5 &= \int_{B_{3\rho/2}} \psi(x) |u(x) - \bar{u}_{B_{3\rho/2}}| \, dx \\
 &\leq \left(\int_{B_{3\rho/2}} |u(x) - \bar{u}_{B_{3\rho/2}}|^t \, dx \right)^{1/t} \left(\int_{B_{3\rho/2}} \psi^{\frac{t}{t-1}}(x) \, dx \right)^{1-1/t} \\
 &\leq C\rho^{n(\frac{1}{t}-\frac{1}{m})+1} \left(\int_{B_{3\rho/2}} |Du(x)|^m \, dx \right)^{1/m} \left(\int_{B_{3\rho/2}} \psi^{\frac{t}{t-1}}(x) \, dx \right)^{1-1/t} \\
 &\leq \varepsilon \int_{B_{3\rho/2}} |Du(x)|^m \, dx \\
 &\quad + C(\varepsilon)\rho^{\frac{m}{m-1}(n(\frac{1}{t}-\frac{1}{m})+1)} \left(\int_{B_{3\rho/2}} \psi^{\frac{t}{t-1}}(x) \, dx \right)^{(m/(m-1))(1-1/t)}
 \end{aligned}$$

holds true with an arbitrary $\varepsilon > 0$. In the same manner, I_6 and I_7 are

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estimated as well. Namely,

$$\begin{aligned}
I_6 &= \int_{B_{3\rho/2}} |u(x)|^{\ell-1} |u(x) - \bar{u}_{B_{3\rho/2}}| \, dx \\
&\leq C \int_{B_{3\rho/2}} |u(x) - \bar{u}_{B_{3\rho/2}}|^\ell \, dx + C \int_{B_{3\rho/2}} |u(x)|^\ell \, dx \\
&\leq C \rho^{\ell(n(\frac{1}{\ell}-\frac{1}{m})+1)} \left(\int_{B_{3\rho/2}} |Du(x)|^m \, dx \right)^{\ell/m-1} \int_{B_{3\rho/2}} |Du(x)|^m \, dx \\
&\quad + C \int_{B_{3\rho/2}} |u(x)|^\ell \, dx
\end{aligned}$$

and

$$\begin{aligned}
I_7 &= \int_{B_{3\rho/2}} |Du(x)|^{\frac{m(\ell-1)}{\ell}} |u(x) - \bar{u}_{B_{3\rho/2}}| \, dx \\
&\leq \left(\int_{B_{3\rho/2}} |u(x) - \bar{u}_{B_{3\rho/2}}|^\ell \, dx \right)^{1/\ell} \left(\int_{B_{3\rho/2}} |Du(x)|^m \, dx \right)^{1-1/\ell} \\
&\leq C \rho^{n(\frac{1}{\ell}-\frac{1}{m})+1} \left(\int_{B_{3\rho/2}} |Du(x)|^m \, dx \right)^{1/m-1/\ell} \int_{B_{3\rho/2}} |Du(x)|^m \, dx.
\end{aligned}$$

At this point we employ the above bounds into (3.7), divide the both

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sides by ρ^n and then use $B_{3\rho/2} \subset B_{2\rho}$ in order to get

$$\begin{aligned}
 & \int_{B_\rho} |Du(x)|^m + |u(x)|^\ell \, dx \\
 & \leq C \left(\int_{B_{2\rho}} (|Du(x)|^m + |u(x)|^\ell)^{\frac{\widehat{m}}{m}} \, dx \right)^{m/\widehat{m}} + C \int_{B_{2\rho}} \varphi(x)^{\frac{m}{m-1}} \, dx \\
 & \quad + C(\varepsilon) \rho^{\frac{m}{m-1}(n(\frac{1}{t}-\frac{1}{m})+1)} \left(\int_{B_{2\rho}} \psi^{\frac{t}{t-1}}(x) \, dx \right)^{(m/(m-1))(1-1/t)-1} \\
 & \quad \times \int_{B_{2\rho}} \psi^{\frac{t}{t-1}}(x) \, dx \\
 & \quad + C \left(\varepsilon + \rho^{\ell(n(\frac{1}{\ell}-\frac{1}{m})+1)} \left(\int_{B_{2\rho}} |Du(x)|^m \, dx \right)^{\ell/m-1} \right. \\
 & \quad \left. + \rho^{n(\frac{1}{\ell}-\frac{1}{m})+1} \left(\int_{B_{2\rho}} |Du(x)|^m \, dx \right)^{1/m-1/\ell} \right) \int_{B_{2\rho}} |Du(x)|^m \, dx.
 \end{aligned} \tag{3.8}$$

Case 2: $B_{3\rho/2} \setminus \Omega \neq \emptyset$. Take $v(x) = u(x)\zeta^2(x)$ as test function in (3.4) with $\zeta \in C_0^\infty(B_{2\rho})$, $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on B_ρ and $|D\zeta| \leq c/\rho$, and use (3.1),

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(3.3) and the properties of ζ to get

$$\begin{aligned}
 \gamma \int_{B_\rho} |Du(x)|^m dx &\leq \Lambda \int_{B_{2\rho}} \varphi(x)^{\frac{m}{m-1}} dx + 2\Lambda \underbrace{\int_{B_{2\rho}} |u(x)|^\ell dx}_{J_1} \\
 &\quad + 2\Lambda \underbrace{\int_{B_{2\rho}} \varphi(x)|u(x)||D\zeta(x)| dx}_{J_2} \\
 &\quad + 2\Lambda \underbrace{\int_{B_{2\rho}} |u(x)|^{\frac{\ell(m-1)}{m}+1}|D\zeta(x)| dx}_{J_3} \\
 &\quad + 2\Lambda \underbrace{\int_{B_{2\rho}} |Du(x)|^{m-1}|u(x)||D\zeta(x)| dx}_{J_4} \\
 &\quad + \Lambda \underbrace{\int_{B_{2\rho}} \psi(x)|u(x)| dx}_{J_5} \\
 &\quad + \Lambda \underbrace{\int_{B_{2\rho}} |Du(x)|^{\frac{m(\ell-1)}{\ell}}|u(x)| dx}_{J_6}.
 \end{aligned} \tag{3.9}$$

We will estimate the terms on the right-hand side of (3.9) by means of the boundary Sobolev inequality (2.3). Precisely,

$$\begin{aligned}
 J_1 &= \int_{B_{2\rho}} |u(x)|^\ell dx \leq C\rho^{\ell(n(\frac{1}{\ell}-\frac{1}{m})+1)} \left(\int_{B_{2\rho}} |Du(x)|^m dx \right)^{\ell/m} \\
 &= C\rho^{\ell(n(\frac{1}{\ell}-\frac{1}{m})+1)} \left(\int_{B_{2\rho}} |Du(x)|^m dx \right)^{\ell/m-1} \int_{B_{2\rho}} |Du(x)|^m dx.
 \end{aligned}$$

Using the Young inequality, (2.3) and taking into account $|D\zeta| \leq c/\rho$, we

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have

$$\begin{aligned}
J_2 &= \int_{B_{2\rho}} \varphi(x) |u(x)| |D\zeta(x)| \, dx \\
&\leq C \int_{B_{2\rho}} |u(x)|^m |D\zeta(x)|^m \, dx + C \int_{B_{2\rho}} \varphi(x)^{\frac{m}{m-1}} \, dx \\
&\leq \frac{C}{\rho^m} \int_{B_{2\rho}} |u(x)|^m \, dx + C \int_{B_{2\rho}} \varphi(x)^{\frac{m}{m-1}} \, dx \\
&\leq C \rho^{n-n\frac{m}{\widehat{m}}} \left(\int_{B_{2\rho}} |Du(x)|^{\widehat{m}} \, dx \right)^{m/\widehat{m}} + C \int_{B_{2\rho}} \varphi(x)^{\frac{m}{m-1}} \, dx
\end{aligned}$$

where $\widehat{m} := \max \left\{ \frac{nm}{n+m}, P \right\}$. Similarly,

$$\begin{aligned}
J_3 &= \int_{B_{2\rho}} |u(x)|^{\frac{\ell(m-1)}{m}+1} |D\zeta(x)| \, dx \leq \frac{C}{\rho^m} \int_{B_{2\rho}} |u(x)|^m \, dx + C \int_{B_{2\rho}} |u(x)|^\ell \, dx \\
&\leq C \rho^{n-n\frac{m}{\widehat{m}}} \left(\int_{B_{2\rho}} |Du(x)|^{\widehat{m}} \, dx \right)^{m/\widehat{m}} + C \int_{B_{2\rho}} |u(x)|^\ell \, dx \\
&= C \rho^{n-n\frac{m}{\widehat{m}}} \left(\int_{B_{2\rho}} |Du(x)|^{\widehat{m}} \, dx \right)^{m/\widehat{m}} + C J_1
\end{aligned}$$

and

$$\begin{aligned}
J_4 &= \int_{B_{2\rho}} |Du(x)|^{m-1} |u(x)| |D\zeta(x)| \, dx \\
&\leq \varepsilon \int_{B_{2\rho}} |Du(x)|^m \, dx + \frac{C(\varepsilon)}{\rho^m} \int_{B_{2\rho}} |u(x)|^m \, dx \\
&\leq \varepsilon \int_{B_{2\rho}} |Du(x)|^m \, dx + C(\varepsilon) \rho^{n-n\frac{m}{\widehat{m}}} \left(\int_{B_{2\rho}} |Du(x)|^{\widehat{m}} \, dx \right)^{m/\widehat{m}}
\end{aligned}$$

with arbitrary $\varepsilon > 0$.

To estimate J_5 , we take the exponent t as above, namely $t = \ell$ if $m < n$ and $t > \max\{\frac{q}{q-1}, m\}$ otherwise, and apply the Hölder inequality, (2.3) and

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the Young inequality. Thus

$$\begin{aligned}
 J_5 &= \int_{B_{2\rho}} \psi(x) |u(x)| \, dx \leq \left(\int_{B_{2\rho}} |u(x)|^t \, dx \right)^{1/t} \left(\int_{B_{2\rho}} \psi^{\frac{t}{t-1}}(x) \, dx \right)^{1-1/t} \\
 &\leq C \rho^{n(\frac{1}{t}-\frac{1}{m})+1} \left(\int_{B_{2\rho}} |Du(x)|^m \, dx \right)^{1/m} \left(\int_{B_{2\rho}} \psi^{\frac{t}{t-1}}(x) \, dx \right)^{1-1/t} \\
 &\leq \varepsilon \int_{B_{2\rho}} |Du(x)|^m \, dx + C(\varepsilon) \rho^{\frac{m}{m-1}(n(\frac{1}{t}-\frac{1}{m})+1)} \left(\int_{B_{2\rho}} \psi^{\frac{t}{t-1}}(x) \, dx \right)^{(m/(m-1))(1-1/t)}
 \end{aligned}$$

holds with arbitrary $\varepsilon > 0$, and the remaining term J_6 is estimated in the same manner

$$\begin{aligned}
 J_6 &= \int_{B_{2\rho}} |Du(x)|^{\frac{m(\ell-1)}{\ell}} |u(x)| \, dx \\
 &\leq \left(\int_{B_{2\rho}} |u(x)|^\ell \, dx \right)^{1/\ell} \left(\int_{B_{2\rho}} |Du(x)|^m \, dx \right)^{1-1/\ell} \\
 &\leq C \rho^{n(\frac{1}{\ell}-\frac{1}{m})+1} \left(\int_{B_{2\rho}} |Du(x)|^m \, dx \right)^{1/m-1/\ell} \int_{B_{2\rho}} |Du(x)|^m \, dx.
 \end{aligned}$$

Using the bounds for $J_1 - J_6$ in (3.9) leads once again to (3.8) with the difference that $\widehat{m} := \max \left\{ \frac{nm}{n+m}, P \right\}$ now. However, a careful analysis of the estimates for the terms $I_1 - I_4$ above shows that these remain valid also with $\widehat{m} := \max \left\{ \frac{nm}{n+m}, P \right\}$ because of $P < m$ and the fact that the Sobolev conjugate of P is anyway *greater than* m if $P > \frac{nm}{n+m}$.

Therefore, (3.8) holds true with $\widehat{m} := \max \left\{ \frac{nm}{n+m}, P \right\}$ in the both cases considered above.

Looking at (3.8) we recall that $n \left(\frac{1}{\ell} - \frac{1}{m} \right) + 1 \geq 0$ and $\frac{\ell}{m} > 1$. Thus, thanks also to the absolute continuity of the Lebesgue integral, we can choose ε and ρ_0 so small that if $\rho < \rho_0$ then the multiplier of $\int_{B_{2\rho}} |Du(x)|^m \, dx$ at the right-hand side of (3.8) becomes less than $1/2$.

To apply Proposition 2.2.4, we consider the functions

$$G(x) = \begin{cases} \left(|Du(x)|^m + |u(x)|^\ell \right)^{\frac{\widehat{m}}{m}} & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega \end{cases}$$

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and

$$F(x) = \begin{cases} \left(C(\varepsilon) \rho_0^{\frac{m}{m-1} \left(n \left(\frac{1}{t} - \frac{1}{m} \right) + 1 \right)} \left(\int_{\Omega} \psi^{\frac{t}{t-1}}(x) dx \right)^{\frac{m}{m-1} \left(1 - \frac{1}{t} \right) - 1} \psi^{\frac{t}{t-1}}(x) \right. \\ \left. + C \varphi(x)^{\frac{m}{m-1}} \right)^{\frac{\hat{m}}{m}} & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega, \end{cases}$$

and set $s = \frac{m}{\hat{m}}$, $s_0 = \kappa \frac{m}{\hat{m}}$ where $\kappa \in \left(1, \min \left\{ \frac{p(m-1)}{m}, \frac{q(t-1)}{t} \right\} \right)$. It is worth noting that the existence of such κ is ensured by our hypotheses

$$p > \frac{m}{m-1}, \quad q > \frac{nm}{nm + m - n}, \quad t > \max \left\{ \frac{q}{q-1}, m \right\}.$$

Moreover, $n \left(\frac{1}{t} - \frac{1}{m} \right) + 1 \geq 0$ and $\frac{m}{m-1} \left(1 - \frac{1}{t} \right) - 1 > 0$.

With these settings, the inequality (3.8) rewrites into

$$\int_{B_\rho} G^s(x) dx \leq C \left(\int_{B_{2\rho}} G(x) dx \right)^s + \int_{B_{2\rho}} F^s(x) dx + \frac{1}{2} \int_{B_{2\rho}} G^s(x) dx$$

for each ball B_ρ with $\rho < \rho_0$ such that $B_{2\rho} \subset B$, where B is a large enough ball containing the bounded domain Ω .

At this point Proposition 2.2.4 applies to ensure existence of exponents $m_0 > m$ and $\ell_0 > \ell$, and a constant C such that

$$\|Du\|_{L^{m_0}(B_\rho)} + \|u\|_{L^{\ell_0}(B_\rho)} \leq C \quad \forall \rho < \rho_0.$$

The desired estimate (3.6), with a constant C depending on known quantities, on $\|Du\|_{L^m(\Omega)}$ and the uniform integrability of $|Du|^m$, then follows by simple covering argument. \square

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3.3 Global Essential Boundedness

3.3.1 Proof of Theorem 3.1.1

Let us start with the case $m < n$. Consider the measure

$$d\mathcal{M} := \left(\chi(x) + \varphi(x)^{\frac{m}{m-1}} + \psi(x) + |u(x)|^{\frac{m^2}{n-m}} \right) dx$$

where $\chi(x)$ is the characteristic function of the domain Ω , dx is the Lebesgue measure and φ and ψ are supposed to be extended as zero outside Ω .

Given a ball B_ρ of radius ρ , we employ the assumptions on φ and ψ to get

$$\begin{aligned} \int_{B_\rho} \varphi(x)^{\frac{m}{m-1}} dx &\leq \|\varphi\|_{L^{p,\lambda}(\Omega)}^{\frac{m}{m-1}} \rho^{n - \frac{m(n-\lambda)}{p(m-1)}} = \|\varphi\|_{L^{p,\lambda}(\Omega)}^{\frac{m}{m-1}} \rho^{n-m + \left(m - \frac{m(n-\lambda)}{p(m-1)}\right)} \\ \int_{B_\rho} \psi(x) dx &\leq \|\psi\|_{L^{q,\mu}(\Omega)} \rho^{n - \frac{n-\mu}{q}} = \|\psi\|_{L^{q,\mu}(\Omega)} \rho^{n-m + \left(m - \frac{n-\mu}{q}\right)} \end{aligned}$$

with $m - \frac{m(n-\lambda)}{p(m-1)} > 0$ and $m - \frac{n-\mu}{q} > 0$ as consequence of the hypotheses $(m-1)p + \lambda > n$ and $mq + \mu > n$.

Further on, $u \in L^{\ell_0}(\Omega)$ by (3.6) and therefore the Hölder inequality gives

$$\int_{B_\rho} |u(x)|^{\frac{m^2}{n-m}} dx \leq \|u\|_{L^{\ell_0}(\Omega)}^{\frac{m^2}{n-m}} \rho^{n-m + \frac{m\ell_0(n-m) - nm^2}{\ell_0(n-m)}} \quad (3.10)$$

with $\frac{m\ell_0(n-m) - nm^2}{\ell_0(n-m)} > 0$ because of $\ell_0 > \ell = \frac{nm}{n-m}$.

This way, setting

$$\varepsilon_0 := \min \left\{ m - \frac{m(n-\lambda)}{p(m-1)}, m - \frac{n-\mu}{q}, \frac{m\ell_0(n-m) - nm^2}{\ell_0(n-m)} \right\} > 0$$

we get

$$\mathcal{M}(B_\rho) \leq K \rho^{n-m+\varepsilon_0}$$

with a constant K depending on known quantities.

For an arbitrary $k \geq 1$, we consider now the function

$$v(x) := \max\{u(x) - k, 0\}$$

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and its upper zero-level set

$$\Omega_k := \{x \in \Omega : u(x) > k\}.$$

It is immediate that $v \equiv 0$ on $\Omega \setminus \Omega_k$ and $v \in W_0^{1,m}(\Omega)$.

The Hölder inequality gives

$$\int_{\Omega} v(x) d\mathcal{M} = \int_{\Omega_k} v(x) d\mathcal{M} \leq \left(\int_{\Omega_k} d\mathcal{M} \right)^{1-1/s} \left(\int_{\Omega_k} |v(x)|^s d\mathcal{M} \right)^{1/s},$$

whence, applying the Adams trace inequality (Proposition 2.2.3) with

$$\alpha_0 = n - m + \varepsilon_0, \quad s = \frac{m(n - m + \varepsilon_0)}{n - m}, \quad r = m,$$

we get

$$\int_{\Omega} v(x) d\mathcal{M} \leq (\mathcal{M}(\Omega_k))^{1 - \frac{n-m}{m(n-m+\varepsilon_0)}} \left(\int_{\Omega_k} |Dv(x)|^m dx \right)^{1/m}. \quad (3.11)$$

To estimate the $L^m(\Omega_k)$ -norm of the gradient Du above, we will apply (3.1) and (3.3). For, the Young inequality implies

$$|\xi|^{\frac{nm-n+m}{n}} |z| \leq \varepsilon |\xi|^m + C(\varepsilon) |z|^{\frac{nm}{n-m}}$$

so that the controlled growth assumptions (3.1) yield

$$b(x, z, \xi)z \leq |z| |b(x, z, \xi)| \leq \Lambda \left(\varepsilon |\xi|^m + C(\varepsilon) |z|^{\frac{nm}{n-m}} + |z| \psi(x) \right)$$

for a.a. $x \in \Omega$, for all $(x, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and with arbitrary $\varepsilon > 0$ to be chosen later. In particular, keeping in mind

$$0 < \frac{u(x) - k}{u(x)} < 1 \quad \text{a.e. } \Omega_k,$$

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we have

$$\begin{aligned} |b(x, u(x), Du(x))v(x)| &= |b(x, u(x), Du(x))u(x)| \frac{u(x) - k}{u(x)} \\ &\leq \Lambda \left(\varepsilon |Du(x)|^m + C(\varepsilon) |u(x)|^{\frac{nm}{n-m}} + |u(x)|\psi(x) \right) \end{aligned} \quad (3.12)$$

for a.a. $x \in \Omega_k$.

At this point, we employ $v \in W_0^{1,m}(\Omega)$ as test function in (3.4) and use $v \equiv 0$ on $\Omega \setminus \Omega_k$, $|Dv| = |Du|$ a.e. Ω_k and (3.3), in order to conclude that

$$\begin{aligned} \int_{\Omega_k} |Dv(x)|^m dx &\leq C \left(\underbrace{\int_{\Omega_k} \varphi(x)^{\frac{m}{m-1}} dx}_{I_1} + \underbrace{\int_{\Omega_k} |u(x)|\psi(x) dx}_{I_2} \right. \\ &\quad \left. + \underbrace{\int_{\Omega_k} |u(x)|^{\frac{nm}{n-m}} dx}_{I_3} \right) \end{aligned} \quad (3.13)$$

after choosing appropriately ε in (3.12).

It is immediate that

$$I_1 = \int_{\Omega_k} \varphi(x)^{\frac{m}{m-1}} dx \leq \mathcal{M}(\Omega_k). \quad (3.14)$$

Further on, we have

$$\begin{aligned} I_2 &= \int_{\Omega_k} |u(x)|\psi(x) dx = \int_{\Omega_k} |u(x) - k + k|\psi(x) dx \\ &\leq \int_{\Omega_k} v(x)\psi(x) dx + k \int_{\Omega_k} \psi(x) dx \\ &\leq \int_{\Omega_k} v(x)\psi(x) dx + k\mathcal{M}(\Omega_k). \end{aligned} \quad (3.15)$$

To estimate the first term on the right-hand side above, define the measure $d\overline{\mathcal{M}} := \psi(x) dx$. We have $\overline{\mathcal{M}}(B_\rho) \leq C(n)\rho^{n-\frac{n-\mu}{q}}$ for each ball B_ρ because of $\psi \in L^{q,\mu}(\Omega)$ and therefore Proposition 2.2.3 can be applied with $s =$

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$\frac{m(n-\frac{n-\mu}{q})}{n-m}$ and $r = m$. Namely,

$$\begin{aligned} \int_{\Omega_k} v(x)\psi(x) dx &= \int_{\Omega_k} v(x) d\overline{\mathcal{M}} \leq \left(\int_{\Omega_k} d\overline{\mathcal{M}} \right)^{1-1/s} \left(\int_{\Omega_k} |v(x)|^s d\overline{\mathcal{M}} \right)^{1/s} \\ &\leq (\overline{\mathcal{M}}(\Omega_k))^{1-1/s} \left(\int_{\Omega_k} |Dv(x)|^m dx \right)^{1/m}. \end{aligned}$$

We use the Young inequality to estimate the last term above by

$$\varepsilon \int_{\Omega_k} |Dv(x)|^m dx + C(\varepsilon) (\overline{\mathcal{M}}(\Omega_k))^{\frac{m}{m-1} \frac{s-1}{s}}$$

with arbitrary $\varepsilon > 0$. Moreover,

$$(\overline{\mathcal{M}}(\Omega_k))^{\frac{m}{m-1} \frac{s-1}{s}} \leq (\mathcal{M}(\Omega_k))^{\frac{m}{m-1} \frac{s-1}{s}} \leq \mathcal{M}(\Omega_k) (\mathcal{M}(\Omega))^{\frac{m}{m-1} \frac{s-1}{s} - 1}$$

and

$$\begin{aligned} \mathcal{M}(\Omega) &= \int_{\Omega} dx + \int_{\Omega} \varphi(x)^{\frac{m}{m-1}} dx + \int_{\Omega} \psi(x) dx + \int_{\Omega} |u(x)|^{\frac{m^2}{n-m}} dx \\ &\leq |\Omega| + C(n, m, p, \lambda, q, \mu, \text{diam } \Omega) \left(\|\varphi\|_{L^{p,\lambda}(\Omega)}^{\frac{m}{m-1}} + \|\psi\|_{L^{q,\mu}(\Omega)} + \|u\|_{L^{\ell_0(\Omega)}(\Omega)}^{\frac{m^2}{n-m}} \right). \end{aligned}$$

Thus, remembering (3.6), $\mathcal{M}(\Omega)$ is bounded in terms of known quantities and $\|Du\|_{L^m(\Omega)}$, whence

$$I_2 \leq \varepsilon \int_{\Omega_k} |Dv(x)|^m dx + C(\varepsilon) k \mathcal{M}(\Omega_k) \quad (3.16)$$

with arbitrary $\varepsilon > 0$.

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In the same manner we estimate also the last term I_3 of (3.13). Precisely,

$$\begin{aligned}
I_3 &= \int_{\Omega_k} |u(x)|^{\frac{nm}{n-m}} dx = \int_{\Omega_k} |u(x)|^m |u(x)|^{\frac{m^2}{n-m}} dx \\
&= \int_{\Omega_k} |u(x) - k + k|^m |u(x)|^{\frac{m^2}{n-m}} dx \\
&\leq 2^{m-1} \left(\int_{\Omega_k} v^m(x) |u(x)|^{\frac{m^2}{n-m}} dx + k^m \int_{\Omega_k} |u(x)|^{\frac{m^2}{n-m}} dx \right) \\
&\leq 2^{m-1} \int_{\Omega_k} v^m(x) |u(x)|^{\frac{m^2}{n-m}} dx + 2^{m-1} k^m \mathcal{M}(\Omega_k).
\end{aligned}$$

We will estimate the first term above with the aid of the Adams trace inequality. For this goal, note that (3.10) implies $|u|^{\frac{m^2}{n-m}} \in L^{1,\theta}(\Omega)$ with

$$\theta = n - m + \frac{m\ell_0(n-m) - nm^2}{\ell_0(n-m)} > n - m.$$

Therefore, there exists an $r' < m$, close enough to m , and such that

$$n - m < \frac{m}{r'}(n - r') < \theta.$$

We have then

$$n - r' + \frac{(n - r')(m - r')}{r'} < \theta$$

and Proposition 2.2.1 yields $|u|^{\frac{m^2}{n-m}} \in L^{1, n-r' + \frac{(n-r')(m-r')}{r'}}(\Omega)$. This way, Proposition 2.2.3 and the Hölder inequality give

$$\begin{aligned}
\int_{\Omega_k} v^m(x) |u(x)|^{\frac{m^2}{n-m}} dx &\leq C \left(\int_{\Omega_k} |Dv(x)|^{r'} dx \right)^{m/r'} \\
&\leq C |\Omega_k|^{\frac{m}{r'}-1} \left(\int_{\Omega_k} |Dv(x)|^m dx \right)
\end{aligned}$$

with C depending also on $\left\| |u|^{\frac{m^2}{n-m}} \right\|_{L^{1,\theta}(\Omega)}$ which is bounded in terms of

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$\|u\|_{L^{\ell_0}(\Omega)}$ (cf. (3.10) and (3.6)). Therefore,

$$I_3 \leq C \left(|\Omega_k|^{\frac{m}{r'}-1} \int_{\Omega_k} |Dv(x)|^m dx + k^m \mathcal{M}(\Omega_k) \right) \quad (3.17)$$

and putting (3.14), (3.16) and (3.17) together, (3.13) takes on the form

$$\int_{\Omega_k} |Dv(x)|^m dx \leq C \left(|\Omega_k|^{\frac{m}{r'}-1} \int_{\Omega_k} |Dv(x)|^m dx + k^m \mathcal{M}(\Omega_k) \right) \quad (3.18)$$

after choosing $\varepsilon > 0$ small enough and remembering $k \geq 1$.

We have further

$$k^{\frac{nm}{n-m}} |\Omega_k| \leq \int_{\Omega_k} |u(x)|^{\frac{nm}{n-m}} dx \leq \int_{\Omega} |u(x)|^{\frac{nm}{n-m}} dx \leq C \|Du\|_{L^m(\Omega)}^{\frac{nm}{n-m}}$$

and this means that if $k \geq k_0$ for large enough k_0 , depending on known quantities and on $\|Du\|_{L^m(\Omega)}$, then the multiplier factor $C|\Omega_k|^{\frac{m}{r'}-1}$ on the right-hand side of (3.18) can be made less than 1/2. This way

$$\int_{\Omega_k} |Dv(x)|^m dx \leq Ck^m \mathcal{M}(\Omega_k) \quad \forall k \geq k_0 \quad (3.19)$$

and then (3.11) becomes

$$\int_{\Omega_k} v(x) d\mathcal{M} \leq Ck (\mathcal{M}(\Omega_k))^{1+\frac{\varepsilon_0}{m(n-m+\varepsilon_0)}} \quad \forall k \geq k_0 \quad (3.20)$$

with $\varepsilon_0 > 0$.

Employing the Cavalieri principle, we have

$$\int_{\Omega_k} v(x) d\mathcal{M} = \int_{\Omega_k} (u(x) - k) d\mathcal{M} = \int_k^\infty \mathcal{M}(\Omega_t) dt$$

and the setting $\tau(t) := \mathcal{M}(\Omega_t)$ rewrites (3.20) into

$$\int_k^\infty \tau(t) dt \leq Ck(\tau(k))^{1+\delta} \quad \forall k \geq k_0, \quad \delta = \frac{\varepsilon_0}{m(n-m+\varepsilon_0)} > 0.$$

It remains to apply the Hartman–Stampacchia maximum principle (Propo-

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sition 2.2.2) to conclude

$$u(x) \leq k_{\max} \quad \text{a.e. } \Omega$$

where k_{\max} depends on known quantities and on $\|Du\|_{L^m(\Omega)}$ in addition.

Repeating the above procedure with $-u(x)$ instead of $u(x)$, we get a bound from below for $u(x)$ which gives the desired estimate (3.5) when $m < n$.

The claim of Theorem 3.1.1 in the limit case $m = n$ can be easily obtained by adapting the above procedure to the new situation. Precisely, the controlled growth condition (3.1) for the term $b(x, z, \xi)$ and the coercivity condition (3.3) have now the form

$$|b(x, z, \xi)| \leq \Lambda \left(\psi(x) + |z|^{\ell-1} + |\xi|^{\frac{n(\ell-1)}{\ell}} \right), \quad (3.21)$$

$$\mathbf{a}(x, z, \xi) \cdot \xi \geq \gamma |\xi|^n - \Lambda |z|^\ell - \Lambda \varphi(x)^{\frac{n}{n-1}}, \quad (3.22)$$

respectively, where $\ell > n$ is an *arbitrary* exponent (cf. (3.2)), $\varphi \in L^{p,\lambda}(\Omega)$ with $p > \frac{n}{n-1}$, $\lambda \in (0, n)$ and $(n-1)p + \lambda > n$, $\psi \in L^{q,\mu}(\Omega)$ with $q > 1$, $\mu \in (0, n)$ and $nq + \mu > n$.

Without loss of generality, we may choose a number $m' < n$, close enough to n , and such that $\ell = \frac{n^2}{(n-m')(n+1)}$. Setting $\ell' = \frac{nm'}{n-m'}$, we have

$$\ell < \ell', \quad \frac{n(\ell-1)}{\ell} = \frac{m'(\ell'-1)}{\ell'}$$

and therefore (3.21) becomes

$$|b(x, z, \xi)| \leq \Lambda \left(\psi(x) + |z|^{\ell'-1} + |\xi|^{\frac{m'(\ell'-1)}{\ell'}} \right) \quad (3.23)$$

for $|z| \geq 1$ and $|\xi| \geq 1$, while (3.22) takes on the form

$$\begin{aligned} \mathbf{a}(x, z, \xi) \cdot \xi &\geq \gamma |\xi|^n - \Lambda |z|^\ell - \Lambda \varphi(x)^{\frac{n}{n-1}} \\ &\geq \gamma |\xi|^{m'} - \Lambda |z|^{\ell'} - \Lambda \varphi(x)^{\frac{m'}{m'-1}} \end{aligned} \quad (3.24)$$

when $|z| \geq 1$ and $|\xi| \geq 1$ and where, without loss of generality, we have supposed $\varphi(x) \geq 1$.

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Defined now the measure

$$d\mathcal{M}' = \left(\chi(x) + \varphi(x)^{\frac{m'}{m'-1}} + \psi(x) + |u(x)|^{\frac{m'^2}{n-m'}} \right) dx,$$

we may increase, if necessary, the value of m' , maintaining it anyway less than n , in order to have $\frac{m'}{m'-1} < p$, $m' > \frac{n-\lambda}{p} + 1$, $m' > \frac{n-\mu}{q}$ and therefore

$$\mathcal{M}'(B_\rho) \leq K\rho^{n-m'+\varepsilon_0}$$

as above, with a suitable $\varepsilon_0 > 0$.

Considering the function $v(x)$ and the sets Ω_k as defined before, it is immediate that

$$\int_{\{x \in \Omega_k : |Dv(x)| < 1\}} |Dv(x)|^{m'} dx \leq |\Omega_k| \leq k^{m'} \mathcal{M}'(\Omega_k),$$

while

$$\int_{\{x \in \Omega_k : |Dv(x)| \geq 1\}} |Dv(x)|^{m'} dx$$

can be estimated with the aid of (3.23) and (3.24) as already did when $m < n$. That leads to the bound (3.19) with m' instead of m and it remains to run the same procedure employed above in order to complete the proof of Theorem 3.1.1. \square

3.3.2 Sharpness of the Hypotheses

We will show, on the level of simple examples built on the m -Laplace operator, that the restrictions on the growths with respect to u and Du and on the Sobolev–Morrey exponents as asked in (3.1) and (3.3) are *sharp* in order to have essential boundedness of the weak solutions to (1.1).

Example 3.3.1 (The $|u|$ -growth $\ell-1$ of $b(x, u, Du)$ is *optimal* for the boundedness.). Let $\varkappa > \ell - 1 > m - 1$. The function

$$u(x) := |x|^{\frac{m}{m-\varkappa-1}} \in W^{1,m}$$

is a local weak solution of the equation

$$\operatorname{div} (|Du|^{m-2} Du) = C(n, m, \varkappa) |u|^\varkappa$$

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in the unit ball $B_1(0)$, but $u \notin L^\infty(B_1)$.

Example 3.3.2 (The gradient growth $\frac{m(\ell-1)}{\ell}$ of $b(x, u, Du)$ is *optimal* for the boundedness.). Let $m < n$ and $\varkappa \in \left(\frac{m(\ell-1)}{\ell}, m\right)$. The function

$$u(x) := |x|^{\frac{m-\varkappa}{m-\varkappa-1}} - 1$$

is a $W_0^{1,m}(B_1)$ weak solution of the Dirichlet problem for the equation

$$\operatorname{div} (|Du|^{m-2} Du) = C(n, m, \varkappa) |Du|^\varkappa,$$

but $u \notin L^\infty(B_1)$.

Example 3.3.3 (The requirements $\varphi \in L^{p,\lambda}(\Omega)$ with $(m-1)p + \lambda > n$ and $\psi \in L^{q,\mu}(\Omega)$ with $mq + \mu > n$ are *sharp* for the boundedness.). Let $B_R = \{x \in \mathbb{R}^n : |x| < R < 1\}$ and consider the functions

$$\varphi(x) := \frac{x}{|x|^m |\log |x||^{m-1}}$$

and

$$\psi(x) := \frac{(m-n) \log |x| - m + 1}{|x|^m |\log |x||^m}.$$

It is immediate to check that $\varphi \in L^{\frac{n}{m-1}}(B_R; \mathbb{R}^n) \subset L^{p', n-(m-1)p'} \forall p' \in (1, \frac{n}{m-1}]$ but $\varphi \notin L^{p', n-(m-1)p'+\varepsilon}(B_R; \mathbb{R}^n) \forall \varepsilon > 0$; and $\psi \in L^{\frac{n}{m}}(B_R) \subset L^{q', n-mq'}(B_R) \forall q' \in (1, \frac{n}{m}]$, but $\psi \notin L^{q', n-mq'+\varepsilon}(B_R) \forall \varepsilon > 0$.

The *unbounded* function

$$u(x) = \log \left(\frac{\log |x|}{\log R} \right)$$

is a $W_0^{1,m}(B_R)$ -weak solution to the homogeneous Dirichlet problems of both

$$\operatorname{div} (|Du|^{m-2} Du - \varphi(x)) = 0$$

and

$$\operatorname{div} (|Du|^{m-2} Du) = \psi(x).$$

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3.4 Global Hölder Continuity

Let us start with the Hölder regularity of the weak solutions in the *interior* of Ω as claimed in Corollary 3.1.2.

Proof of Corollary 3.1.2. Let $m = n$. Then Lemma 3.2.1 implies $u \in W^{1,m_0}(\Omega)$ with $m_0 > n$ and thus the interior Hölder continuity of u with exponent $1 - \frac{n}{m_0}$ follows from the Morrey lemma.

Suppose therefore $m < n$. We have then $\ell = \frac{nm}{n-m}$ (cf. (3.2)) and, taking into account the essential boundedness of u given by Theorem 3.1.1, the structure conditions (3.1) and (3.3) can be rewritten as

$$\begin{aligned} |\mathbf{a}(x, z, \xi)| &\leq \Lambda \left(\varphi'(x) + |\xi|^{m-1} \right), \\ |b(x, z, \xi)| &\leq \Lambda \left(\psi'(x) + |\xi|^{\frac{m(\ell-1)}{\ell}} \right) \leq \Lambda \left(\psi'(x) + |\xi|^m \right), \\ \mathbf{a}(x, z, \xi) \cdot \xi &\geq \gamma |\xi|^m - \varphi''(x) \end{aligned}$$

for a.a. $x \in \Omega$ and all $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$, where

$$\begin{aligned} \varphi'(x) &= \varphi(x) + M^{\frac{n(m-1)}{n-m}}, \quad \psi'(x) = \psi(x) + M^{\frac{nm-n+m}{n-m}} + 1, \\ \varphi''(x) &= \Lambda \left(M^{\frac{nm}{n-m}} + \varphi(x)^{\frac{m}{m-1}} \right). \end{aligned}$$

Straightforward calculations, based on the hypotheses $\varphi \in L^{p,\lambda}(\Omega)$, $p > \frac{m}{m-1}$, $(m-1)p + \lambda > n$ and $\psi \in L^{q,\mu}(\Omega)$, $q > \frac{mn}{mn+m-n}$, $mq + \mu > n$, give

$$\int_{B_\rho} \varphi'(x) \, dx \leq C \left(n, m, \|\varphi\|_{L^{p,\lambda}(\Omega)}, M, \text{diam } \Omega \right) \rho^{n - \frac{n-\lambda}{p}} = C \rho^{n-m+1+\varepsilon_1}$$

with $\varepsilon_1 = m - 1 - \frac{n-\lambda}{p} > 0$,

$$\int_{B_\rho} \psi'(x) \, dx \leq C \left(n, m, \|\psi\|_{L^{q,\mu}(\Omega)}, M, \text{diam } \Omega \right) \rho^{n - \frac{n-\mu}{q}} = C \rho^{n-m+\varepsilon_2}$$

with $\varepsilon_2 = m - \frac{n-\mu}{q} > 0$, and

$$\int_{B_\rho} \varphi''(x) \, dx \leq C \left(n, m, \Lambda, \|\varphi\|_{L^{p,\lambda}(\Omega)}, M, \text{diam } \Omega \right) \rho^{n - \frac{m(n-\lambda)}{p(m-1)}} = C \rho^{n-m+\varepsilon_3}$$

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with $\varepsilon_3 = m - \frac{m(n-\lambda)}{p(m-1)} > 0$.

At this point, the claim of Corollary 3.1.2 follows from the Harnack inequality proved by Lieberman (see [24, Theorem 4.1] and [37, Theorem 2.2]) and standard covering arguments. \square

To proceed further with the more delicate question of Hölder continuity *up to the boundary* of Ω , we need the following result ensuring suitable growth estimate for the gradient over small balls.

Lemma 3.4.1. *Assume (3.1) and (3.3), and let u be a weak solution to the problem (1.1) extended as zero outside Ω .*

Let B_ρ be a ball of radius $\rho \in (0, \text{diam } \Omega)$ and centered at a point of $\partial\Omega$, and $\eta \in C_0^\infty(B_{\rho/2})$ with $|D\eta| \leq c/\rho$. There exists a constant C depending on the same quantities as M in (3.5), such that

$$\int_{B_{\rho/2}} |D(\eta w^{-1})|^m dx \leq C(M(\rho) + A(\rho)) \left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho) \right)^{m-1} \rho^{n-m}$$

where $M(\rho) = \text{ess sup}_{B_\rho} \bar{u}$, $\bar{u} = \max\{u, 0\}$, $A(\rho) = \rho + \|\varphi\|_{L^{p,\lambda}(B_\rho)}^{\frac{1}{m-1}} + \|\psi\|_{L^{q,\mu}(B_\rho)}^{\frac{1}{m}}$ and $w^{-1} = M(\rho) + A(\rho) - \bar{u}$.

Proof. Let $1 \leq \theta < \min \left\{ \frac{p(m-1)}{m}, q \right\}$ be arbitrary and consider the measure

$$d\mathcal{M}_\theta := \left(\varphi(x)^{\frac{\theta m}{m-1}} + \psi(x)^\theta \right) dx$$

where, as before, φ and ψ are supposed to be extended as zero outside Ω .

Given a ball B_ρ of radius ρ , we have

$$\begin{aligned} \int_{B_\rho} \varphi(x)^{\frac{\theta m}{m-1}} dx &\leq \|\varphi\|_{L^{p,\lambda}(B_\rho)}^{\frac{\theta m}{m-1}} \rho^{n - \frac{\theta m(n-\lambda)}{p(m-1)}} \leq A(\rho)^{\theta m} \rho^{n - \frac{\theta m(n-\lambda)}{p(m-1)}} \\ \int_{B_\rho} \psi(x)^\theta dx &\leq \|\psi\|_{L^{q,\mu}(B_\rho)}^\theta \rho^{n - \frac{\theta(n-\mu)}{q}} \leq A(\rho)^{\theta m} \rho^{n - \frac{\theta(n-\mu)}{q}}. \end{aligned}$$

Therefore, setting

$$\alpha_\theta := \min \left\{ n - \frac{\theta m(n-\lambda)}{p(m-1)}, n - \frac{\theta(n-\mu)}{q} \right\} > 0, \quad (3.25)$$

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we get

$$\mathcal{M}_\theta(B_\rho) \leq C A(\rho)^{\theta m} \rho^{\alpha_\theta}$$

with a constant C depending on known quantities.

Now, let B_ρ be a ball of radius ρ centered at a boundary point and use $v := \eta^m e^{\frac{\Lambda}{\gamma} \bar{u}} v_0$ as test function in (3.4), where $\eta \in C_0^\infty(B_\rho)$, $v_0 = w^\beta - (M(\rho) + A(\rho))^{-\beta}$ and $\beta > 0$ is a parameter under control. Having in mind that $|D\bar{u}| = |Du|$ a.e. $\{x \in \Omega: u(x) > 0\}$, we obtain

$$\begin{aligned} & \int \eta^m e^{\frac{\Lambda}{\gamma} \bar{u}} \left(\beta w^{\beta+1} + \frac{\Lambda}{\gamma} v_0 \right) \mathbf{a}(x, u(x), Du(x)) \cdot Du \, dx \\ & + \int m \eta^{m-1} e^{\frac{\Lambda}{\gamma} \bar{u}} v_0 \mathbf{a}(x, u(x), Du(x)) \cdot D\eta \, dx + \int b(x, u(x), Du(x)) v \, dx = 0 \end{aligned}$$

and hereafter all the integrals will be taken over $B_\rho \cap \{x \in \Omega: u(x) > 0\}$. Using that

$$|Du|^{\frac{m(\ell-1)}{\ell}} \leq |Du|^m + 1$$

in view of the Young inequality, we get

$$\begin{aligned} \gamma \beta \int \eta^m e^{\frac{\Lambda}{\gamma} \bar{u}} w^{\beta+1} |Du|^m \, dx & \leq \Lambda m \int \eta^{m-1} e^{\frac{\Lambda}{\gamma} \bar{u}} v_0 \left(|Du|^{m-1} + |u|^{\frac{\ell(m-1)}{m}} + \varphi \right) |D\eta| \, dx \\ & + \Lambda \int \eta^m e^{\frac{\Lambda}{\gamma} \bar{u}} v_0 (|u|^{\ell-1} + \psi + 1) \, dx \\ & + \Lambda \int \eta^m e^{\frac{\Lambda}{\gamma} \bar{u}} \left(\beta w^{\beta+1} + \frac{\Lambda}{\gamma} v_0 \right) \left(|u|^\ell + \varphi^{\frac{m}{m-1}} \right) \, dx \end{aligned}$$

as consequence of (3.1) and (3.3). Since $v_0 \leq w^\beta$ and

$$w^{-1} \leq C \left(\|u\|_{L^\infty(\Omega)} + \|\varphi\|_{L^{p,\lambda}(\Omega)}^{\frac{1}{m-1}} + \|\psi\|_{L^{q,\mu}(\Omega)}^{\frac{1}{m}} + \text{diam } \Omega \right),$$

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it follows

$$\begin{aligned}
 \beta \int \eta^m w^{\beta+1} |Du|^m dx &\leq C \int \eta^{m-1} w^\beta |Du|^{m-1} |D\eta| dx \\
 &+ C \int \eta^{m-1} w^\beta (\varphi + 1) |D\eta| dx \\
 &+ C \int \eta^m w^{\beta+1} (\psi + 1) dx \\
 &+ C(1 + \beta) \int \eta^m w^{\beta+1} \left(\varphi^{\frac{m}{m-1}} + 1 \right) dx
 \end{aligned} \tag{3.26}$$

where the constant C depends on known quantities and on $\|Du\|_{L^m(\Omega)}$ through M in (3.5). We apply the Young inequality to get

$$\begin{aligned}
 \int \eta^{m-1} w^\beta |Du|^{m-1} |D\eta| dx &\leq \varepsilon \int \eta^m w^{\beta+1} |Du|^m dx \\
 &+ C\varepsilon^{1-m} \int w^{\beta-m+1} |D\eta|^m dx \\
 \int \eta^{m-1} w^\beta (\varphi + 1) |D\eta| dx &\leq \beta \int \eta^m w^{\beta+1} \left(\varphi^{\frac{m}{m-1}} + 1 \right) dx \\
 &+ C\beta^{1-m} \int w^{\beta-m+1} |D\eta|^m dx
 \end{aligned}$$

for any $\varepsilon > 0$. Choosing $\varepsilon = \frac{\beta}{2C}$ with appropriate C above, we obtain from (3.26)

$$\begin{aligned}
 \beta \int \eta^m w^{\beta+1} |Du|^m dx &\leq C(1 + \beta) \int \eta^m w^{\beta+1} dx \\
 &+ C(1 + \beta) \int \eta^m w^{\beta+1} \left(\varphi^{\frac{m}{m-1}} + \psi \right) dx \\
 &+ C\beta^{1-m} \int w^{\beta-m+1} |D\eta|^m dx.
 \end{aligned} \tag{3.27}$$

Take now $\beta = m - 1$ in (3.27). We have $M(\rho) - \bar{u} \geq 0$ whence $w \leq A(\rho)^{-1}$ and the Poincaré inequality yields

$$\int \eta^m w^m dx \leq A(\rho)^{-m} \int \eta^m dx \leq C \int |D\eta|^m dx.$$

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Further on,

$$\int \eta^m w^m \left(\varphi^{\frac{m}{m-1}} + \psi \right) dx = \int \eta^m w^m d\mathcal{M}_1 \leq A(\rho)^{-m} \int \eta^m d\mathcal{M}_1.$$

To estimate the term on the right-hand side above, we will distinguish between the cases $m < n$ and $m = n$. Thus, if $m < n$ we apply the Adams trace inequality from Proposition 2.2.3 with

$$\alpha_0 = \alpha_1, \quad s = \frac{\alpha_1 m}{n - m}, \quad r = m,$$

and where α_1 is taken from (3.25) with $\theta = 1$. We have

$$\begin{aligned} A(\rho)^{-m} \int \eta^m d\mathcal{M}_1 &\leq A(\rho)^{-m} \left(\int \eta^s d\mathcal{M}_1 \right)^{m/s} \left(\int d\mathcal{M}_1 \right)^{1-m/s} \\ &\leq A(\rho)^{-m} \left(C A(\rho)^{\frac{m^2}{s}} \int |D\eta|^m dx \right) (C A(\rho)^m \rho^{\alpha_1})^{1-m/s} \\ &\leq C \rho^{\alpha_1 - n + m} \int |D\eta|^m dx \\ &\leq C \int |D\eta|^m dx, \end{aligned}$$

where $\alpha_1 - n + m > 0$ and $0 < \rho < \text{diam } \Omega$ have been used in the last bound.

If instead $m = n$, we employ once again Proposition 2.2.3, but

$$\alpha_0 = \alpha_1, \quad s = n, \quad r = \frac{n^2}{\alpha_1 + n}$$

now. Thus

$$\begin{aligned} A(\rho)^{-n} \int \eta^n d\mathcal{M}_1 &\leq C \left(\int |D\eta|^r dx \right)^{n/r} \leq C \left(\int |D\eta|^n dx \right) \left(\int dx \right)^{n/r-1} \\ &\leq C \rho^{\alpha_1} \int |D\eta|^n dx \\ &\leq C \int |D\eta|^n dx \end{aligned}$$

thanks to $\alpha_1 > 0$ and $0 < \rho < \text{diam } \Omega$.

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This way,

$$\int \eta^m w^m \left(\varphi^{\frac{m}{m-1}} + \psi \right) dx \leq C \int |D\eta|^m dx$$

and (3.27) with $\beta = m - 1$ becomes

$$\int \eta^m |D(\log w)|^m dx \leq C \int |D\eta|^m dx$$

for each $0 \leq \eta \in C_0^\infty(B_\rho)$.

Choosing appropriately η , we are in a position to apply Proposition 2.2.5 that asserts existence of constants C and $\sigma_0 > 0$ such that

$$\int_{B_{3\rho/4}} w^{-\sigma} dx \int_{B_{3\rho/4}} w^\sigma dx \leq C\rho^{2n} \quad \text{for all } |\sigma| \leq \sigma_0. \quad (3.28)$$

Consider now the cases $\beta \neq m - 1$ in (3.27). For, we multiply the both sides of (3.27) by β^{m-1} which rewrites it as

$$\begin{aligned} \beta^m \int \eta^m w^{\beta+1} |Du|^m dx &\leq C \int w^{\beta-m+1} |D\eta|^m dx \\ &+ C(1 + \beta^m) \int \eta^m w^{\beta+1} \left(\varphi^{\frac{m}{m-1}} + \psi + 1 \right) dx. \end{aligned} \quad (3.29)$$

Setting $\beta = mt + m - 1 > 0$, we have

$$|D(\eta w^t)|^m \leq 2^{m-1} (w^{mt} |D\eta|^m + |t|^m \eta^m w^{mt+m} |Du|^m)$$

and the use of (3.29) with $\beta = mt + m - 1$ gives

$$\begin{aligned} \int |D(\eta w^t)|^m dx &\leq C \left(1 + \frac{|t|^m}{(mt + m - 1)^m} \right) \int w^{mt} |D\eta|^m dx \\ &+ C \left(|t|^m + \frac{|t|^m}{(mt + m - 1)^m} \right) \int \eta^m w^{mt+m} \left(\varphi^{\frac{m}{m-1}} + \psi + 1 \right) dx. \end{aligned}$$

We have

$$\frac{t}{mt + m - 1} < \frac{1}{m} \quad \forall t > 0,$$

while $|t|^m < \left(\frac{m-1}{m} \right)^m$ and $\frac{|t|^m}{(mt+m-1)^m}$ is a positive and decreasing function

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whenever $t \in (\frac{1-m}{m}, 0)$.

Thus, defining

$$N(t) := \begin{cases} 1 + t^m & \text{if } t > 0, \\ 1 + \frac{|t|^m}{(mt + m - 1)^m} & \text{if } \frac{1-m}{m} < t \leq 0, \end{cases}$$

the last bound takes on the form

$$\begin{aligned} \int |D(\eta w^t)|^m dx &\leq CN(t) \left(\int w^{mt} |D\eta|^m dx \right. \\ &\quad \left. + \int \eta^m w^{mt+m} \left(\varphi^{\frac{m}{m-1}} + \psi + 1 \right) dx \right). \end{aligned} \quad (3.30)$$

The Young inequality and the definition of the measure \mathcal{M}_θ yield

$$\begin{aligned} \int \eta^m w^{mt+m} \left(\varphi^{\frac{m}{m-1}} + \psi \right) dx &\leq \varepsilon \int \eta^m w^{mt+m} \left(\varphi^{\frac{\theta m}{m-1}} + \psi^\theta \right) dx \\ &\quad + C\varepsilon^{\frac{1}{1-\theta}} \int \eta^m w^{mt+m} dx \\ &= \varepsilon \int \eta^m w^{mt+m} d\mathcal{M}_\theta + C\varepsilon^{\frac{1}{1-\theta}} \int \eta^m w^{mt+m} dx, \end{aligned} \quad (3.31)$$

where θ is such that $1 < \theta < \min \left\{ \frac{p(m-1)}{n-\lambda}, \frac{mq}{n-\mu} \right\}$. In order to estimate the first term on the right-hand side of (3.31) we will employ once again Proposition 2.2.3 distinguishing between the cases $m < n$ and $m = n$.

Let $m < n$. Taking

$$\alpha_0 = \alpha_\theta, \quad s = \frac{\alpha_\theta m}{n - m}, \quad r = m$$

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in Proposition 2.2.3 and remembering $w \leq A(\rho)^{-1}$, we get

$$\begin{aligned}
 \int \eta^m w^{mt+m} d\mathcal{M}_\theta &\leq A(\rho)^{-m} \int (\eta w^t)^m d\mathcal{M}_\theta \\
 &\leq A(\rho)^{-m} \left(\int (\eta w^t)^s d\mathcal{M}_\theta \right)^{m/s} \left(\int d\mathcal{M}_\theta \right)^{1-m/s} \\
 &\leq CA(\rho)^{(\theta-1)m+\alpha_\theta-(n-m)} \int |D(\eta w^t)|^m dx \\
 &\leq C \int |D(\eta w^t)|^m dx
 \end{aligned}$$

since $\theta > 1$ and $\alpha_\theta > n - m$.

In case $m = n$, the application of the Adams trace inequality with

$$\alpha_0 = \alpha_\theta, \quad s = n, \quad r = \frac{n^2}{\alpha_\theta + n}$$

gives

$$\begin{aligned}
 \int \eta^n w^{nt+n} d\mathcal{M}_\theta &\leq A(\rho)^{-n} \int (\eta w^t)^n d\mathcal{M}_\theta \\
 &\leq CA(\rho)^{(\theta-1)n} \left(\int |D(\eta w^t)|^r dx \right)^{n/r} \\
 &\leq CA(\rho)^{(\theta-1)n} \left(\int |D(\eta w^t)|^n dx \right) \left(\int dx \right)^{n/r-1} \\
 &= CA(\rho)^{(\theta-1)n+\alpha_\theta} \int |D(\eta w^t)|^n dx \\
 &\leq C \int |D(\eta w^t)|^n dx.
 \end{aligned}$$

Therefore

$$\int \eta^m w^{mt+m} d\mathcal{M}_\theta \leq C \int |D(\eta w^t)|^m dx$$

and (3.31) becomes

$$\int \eta^m w^{mt+m} \left(\varphi^{\frac{m}{m-1}} + \psi \right) dx \leq C\varepsilon \int |D(\eta w^t)|^m dx + C\varepsilon^{\frac{1}{1-\theta}} \int \eta^m w^{mt+m} dx$$

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with $\varepsilon > 0$ under control. In particular, choosing $\varepsilon = \frac{1}{2CN(t)}$ above and having in mind $N(t) \geq 1$, we get from (3.30)

$$\int |D(\eta w^t)|^m dx \leq CK(t) \int w^{mt} (\eta^m w^m + |D\eta|^m) dx \quad (3.32)$$

with

$$K(t) := (N(t))^{\frac{\theta}{\theta-1}}.$$

Let us take now a cut-off function $\eta \in C_0^\infty(B_r)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ on B_s and $|D\eta| \leq \frac{c}{r-s}$ where $0 < s < r \leq \rho$. Employing the Sobolev inequality, we get

$$\begin{aligned} \left(\int_{B_s} (w^t)^\ell dx \right)^{m/\ell} &= \left(\int_{B_s} (\eta w^t)^\ell dx \right)^{m/\ell} \\ &\leq C \left(\int_{B_r} |D(\eta w^t)|^{\frac{\ell n}{\ell+n}} dx \right)^{m(\ell+n)/(\ell n)} \\ &\leq C \left(\int_{B_r} |D(\eta w^t)|^m dx \right) \left(\int_{B_r} dx \right)^{m(\ell+n)/(\ell n)-1} \\ &\leq \frac{CK(t)}{(r-s)^m} \rho^{\frac{m(\ell+n)-\ell n}{\ell}} \int_{B_r} w^{mt} dx \end{aligned} \quad (3.33)$$

from (3.32) since $\eta w \leq \frac{1}{\rho} \leq \frac{1}{r-s}$.

Let $\rho_k = \rho \left(\frac{1}{2} + \frac{1}{2^{k+2}} \right)$ for $k = 0, 1, \dots$ and let $t_0 > 0$ be any number such that $mt_0 \leq \sigma_0$ with σ_0 appearing in (3.28). Making use of the simple inequalities $e^{2mt} \geq (1+t)^{2m} \geq 1+t^{2m}$ valid for all $t \geq 0$ and all $m \geq \frac{1}{2}$, and remembering the properties of the function $K(t)$, we have

$$K(t) = (1+t^m)^{\frac{\theta}{\theta-1}} \leq e^{\frac{2m\theta}{\theta-1}\sqrt{t}} \quad \forall t > 0.$$

Thus, taking $t = t_0 \left(\frac{\ell}{m} \right)^k > 0$ and using (3.33) with $s = \rho_{k+1}$ and $r = \rho_k$, we

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get

$$\begin{aligned} & \left(\int_{B_{\rho_{k+1}}} w^{mt_0 \left(\frac{\ell}{m}\right)^{k+1}} dx \right)^{(m/\ell)^{k+1}} \\ & \leq (2^k C)^{m \left(\frac{m}{\ell}\right)^k} \rho^{\frac{n(m-\ell)}{\ell} \left(\frac{m}{\ell}\right)^k} e^{\frac{2m\theta}{\theta-1} \sqrt{t_0} \left(\frac{m}{\ell}\right)^{\frac{k}{2}}} \left(\int_{B_{\rho_k}} w^{mt_0 \left(\frac{\ell}{m}\right)^k} dx \right)^{(m/\ell)^k} \end{aligned}$$

for $k = 0, 1, \dots$. Iteration of these inequalities from 0 to $N \in \mathbb{N}$ yields

$$\begin{aligned} & \left(\int_{B_{\rho_{N+1}}} w^{mt_0 \left(\frac{\ell}{m}\right)^{N+1}} dx \right)^{(m/\ell)^{N+1}} \\ & \leq 2^m \sum_{k=0}^N k \left(\frac{m}{\ell}\right)^k C^m \sum_{k=0}^N \left(\frac{m}{\ell}\right)^k e^{\frac{2m\theta}{\theta-1} \sqrt{t_0} \sum_{k=0}^N \left(\frac{m}{\ell}\right)^{\frac{k}{2}}} \rho^{\frac{n(m-\ell)}{\ell} \sum_{k=0}^N \left(\frac{m}{\ell}\right)^k} \int_{B_{\rho_0}} w^{mt_0} dx \end{aligned}$$

and passage to the limit as $N \rightarrow +\infty$ gives

$$\operatorname{ess\,sup}_{B_{\rho/2}} w^{mt_0} = \left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho) \right)^{-mt_0} \leq C \rho^{-n} \int_{B_{3\rho/4}} w^{mt_0} dx. \quad (3.34)$$

This way, it follows from (3.28) and (3.34) that

$$\rho^{-n} \int_{B_{3\rho/4}} w^{-mt_0} dx \leq C \left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho) \right)^{mt_0} \quad (3.35)$$

for any $t_0 > 0$ such that $0 < mt_0 \leq \sigma_0$.

To proceed further, we set $\sigma = -mt_1$ where $t_1 = t_1(m, n) < 0$ will be chosen in the sequel. Let $0 < \sigma < m - 1$ and define $\sigma_1 = \sigma \left(\frac{m}{\ell}\right)^\kappa$ where κ is a positive integer for which $m - 1 \leq \sigma_0 \left(\frac{\ell}{m}\right)^\kappa$ and σ_0 is taken from (3.28). Obviously, $0 < \sigma_1 \left(\frac{\ell}{m}\right)^k \leq \sigma$ for $0 \leq k \leq \kappa$. Thus $\frac{1-m}{m} < -\frac{\sigma}{m} \leq -\frac{\sigma_1}{m} \left(\frac{\ell}{m}\right)^k$ and therefore

$$K \left(-\frac{\sigma_1}{m} \left(\frac{\ell}{m}\right)^k \right) \leq K \left(-\frac{\sigma}{m} \right)$$

for $0 \leq k \leq \kappa$ since K is a decreasing function on $\left(\frac{1-m}{m}, 0\right)$.

We take now $\rho_k = \frac{\rho}{4} \left(3 - \frac{k}{\kappa+1}\right)$ for $0 \leq k \leq \kappa + 1$ and apply (3.33) with

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$s = \rho_{k+1}$, $r = \rho_k$ and $t = -\frac{\sigma_1}{m} \left(\frac{\ell}{m}\right)^k$ in order to get

$$\begin{aligned} & \left(\int_{B_{\rho_{k+1}}} w^{-\sigma_1 \left(\frac{\ell}{m}\right)^{k+1}} dx \right)^{(m/\ell)^{k+1}} \\ & \leq \left(CK \left(-\frac{\sigma}{m} \right) 4^m (\kappa + 1)^m \rho^{\frac{n(m-\ell)}{\ell}} \right)^{\left(\frac{m}{\ell}\right)^k} \left(\int_{B_{\rho_k}} w^{-\sigma_1 \left(\frac{\ell}{m}\right)^k} dx \right)^{(m/\ell)^k} \end{aligned}$$

for $0 \leq k \leq \kappa$. Iteration of these inequalities for $0 \leq k \leq \kappa$ gives

$$\begin{aligned} & \left(\int_{B_{\rho/2}} w^{-\sigma_1 \left(\frac{\ell}{m}\right)^{\kappa+1}} dx \right)^{(m/\ell)^{\kappa+1}} \\ & \leq \left(CK \left(-\frac{\sigma}{m} \right) 4^m (\kappa + 1)^m \right)^{\sum_{k=0}^{\kappa} \left(\frac{m}{\ell}\right)^k} \rho^{\frac{n(m-\ell)}{\ell} \sum_{k=0}^{\kappa} \left(\frac{m}{\ell}\right)^k} \int_{B_{3\rho/4}} w^{-\sigma_1} dx, \end{aligned}$$

whence

$$\rho^{-n} \int_{B_{\rho/2}} w^{-\sigma_1 \left(\frac{\ell}{m}\right)^{\kappa+1}} dx \leq C \left(\rho^{-n} \int_{B_{3\rho/4}} w^{-\sigma_1} dx \right)^{(\ell/m)^{\kappa+1}}.$$

Remembering $0 < \sigma_1 < \sigma_0$, we obtain from (3.35) with $\sigma_1 = mt_0$ that

$$\rho^{-n} \int_{B_{\rho/2}} w^{t_1 \ell} dx \leq C \left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho) \right)^{-t_1 \ell} \quad (3.36)$$

for each $t_1 < 0$ such that $0 < -mt_1 < m - 1$.

Take now $v = \eta^m e^{\frac{\Lambda}{\gamma} \bar{u}} \bar{u}$ as a test function in (3.4). Keeping in mind (3.1), (3.3) and (3.5), we obtain

$$\begin{aligned} \int \eta^m |Du|^m dx & \leq CM(\rho) \int \eta^{m-1} |Du|^{m-1} |D\eta| dx \\ & + CM(\rho) \int \eta^{m-1} |D\eta| (1 + \varphi) dx \\ & + C \int \eta^m \left(\varphi^{\frac{m}{m-1}} + \psi \right) dx + C \int \eta^m dx. \end{aligned} \quad (3.37)$$

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Fix $\eta \in C_0^\infty(B_{\rho/2})$ such that $0 \leq \eta \leq 1$ and $|D\eta| \leq c/\rho$. To estimate the terms on the right-hand side in (3.37), we take a $t_2 < 0$ such that $1 < (1+t_2)m < \frac{\ell}{m}$. Thus

$$\begin{aligned} \int \eta^{m-1} |Du|^{m-1} |D\eta| \, dx &= \int (\eta w^{1+t_2} |Du|)^{m-1} (w^{-(1+t_2)(m-1)} |D\eta|) \, dx \\ &= C \int (\eta |D(w^{t_2})|)^{m-1} (w^{-(1+t_2)(m-1)} |D\eta|) \, dx \\ &\leq C \left(\int (\eta |D(w^{t_2})|)^m \, dx \right)^{1-1/m} \left(\int (w^{-(1+t_2)(m-1)} |D\eta|)^m \, dx \right)^{1/m}. \end{aligned}$$

The two terms above will be estimated with the aid of (3.32) and (3.36), respectively. Precisely, applying (3.32) with $t = t_2$, we have

$$\begin{aligned} \left(\int (\eta |D(w^{t_2})|)^m \, dx \right)^{1-1/m} &\leq C \left(\int (|D(\eta w^{t_2})|)^m \, dx + \int (w^{t_2} |D\eta|)^m \, dx \right)^{1-1/m} \\ &\leq C \left(\left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho) \right)^{-mt_2} \rho^{n-m} \right)^{1-1/m} \end{aligned}$$

since $w \leq (M(\rho) - M(\frac{\rho}{2}) + A(\rho))^{-1}$ on $B_{\rho/2}$ as well as $w \leq A(\rho)^{-1} \leq \rho^{-1}$.

In the same manner, (3.36) with $t_1 = -\frac{(1+t_2)(m-1)m}{\ell}$ leads to

$$\begin{aligned} \left(\int (w^{-(1+t_2)(m-1)} |D\eta|)^m \, dx \right)^{1/m} \\ \leq C \left(\left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho) \right)^{(1+t_2)(m-1)m} \rho^{n-m} \right)^{1/m} \end{aligned}$$

whence

$$\int \eta^{m-1} |Du|^{m-1} |D\eta| \, dx \leq C \left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho) \right)^{m-1} \rho^{n-m}.$$

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Further on,

$$\begin{aligned}
\int \eta^{m-1} |D\eta| (1 + \varphi) \, dx &\leq C \left(\int \eta^m \left(1 + \varphi^{\frac{m}{m-1}} \right) \, dx \right)^{1-1/m} \left(\int |D\eta|^m \, dx \right)^{1/m} \\
&\leq C \left(\rho^n + \mathcal{M}_1(B_{\rho/2}) \right)^{\frac{m-1}{m}} \rho^{\frac{n-m}{m}} \\
&\leq C \left(\rho^m \rho^{n-m} + A(\rho)^m \rho^{\alpha_1} \right)^{\frac{m-1}{m}} \rho^{\frac{n-m}{m}} \\
&\leq C \left(A(\rho)^m \rho^{n-m} \right)^{\frac{m-1}{m}} \rho^{\frac{n-m}{m}} \\
&= C A(\rho)^{m-1} \rho^{n-m}
\end{aligned}$$

because of $\mathcal{M}_1(B_{\rho/2}) \leq A(\rho)^m \rho^{\alpha_1}$ (cf. (3.25)), $\rho \leq A(\rho)$ and $\alpha_1 > n - m$.

Similarly,

$$\int \eta^m \left(\varphi^{\frac{m}{m-1}} + \psi \right) \, dx = \int \eta^m \, d\mathcal{M}_1 \leq C A(\rho)^m \rho^{n-m}$$

and

$$\int \eta^m \, dx = C \rho^m \int |D\eta|^m \, dx \leq C A(\rho)^m \rho^{n-m}$$

in view of the Poincaré inequality.

Therefore, (3.37) yields

$$\int \eta^m |Du|^m \, dx \leq C (M(\rho) + A(\rho)) \left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho) \right)^{m-1} \rho^{n-m}. \quad (3.38)$$

On the other hand,

$$\int |D(\eta w^{-1})|^m \leq C \left(\int w^{-m} |D\eta|^m \, dx + \int \eta^m |Du|^m \, dx \right)$$

and, keeping in mind $w^{-1} \leq M(\rho) + A(\rho)$, we apply (3.36) with $t_1 = \frac{1-m}{\ell}$ to

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get

$$\begin{aligned}
\int w^{-m} |D\eta|^m dx &\leq C\rho^{-m} \int w^{-1} w^{1-m} dx \\
&\leq C(M(\rho) + A(\rho))\rho^{-m} \int w^{1-m} dx \\
&\leq C(M(\rho) + A(\rho)) \left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho) \right)^{m-1} \rho^{n-m}.
\end{aligned}$$

The last bound, together with (3.38) gives

$$\int |D(\eta w^{-1})|^m dx \leq C(M(\rho) + A(\rho)) \left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho) \right)^{m-1} \rho^{n-m}$$

and this completes the proof of Lemma 3.4.1. \square

Once having the result of Lemma 3.4.1 it is easy to extend the Hölder continuity of the weak solutions up to the boundary of Ω , thanks of the P -thickness condition (2.1).

3.4.1 Proof of Theorem 3.1.3

Let $x_0 \in \partial\Omega$ be any point and set $B_\rho = B_\rho(x_0)$ for the sake of brevity. Since $(m-1)p + \lambda > n$ and $mq + \mu > n$, there exist positive constants $\bar{\lambda}$ and $\bar{\mu}$ such that $n < (m-1)p + \bar{\lambda} < (m-1)p + \lambda$, $n < mq + \bar{\mu} < mq + \mu$. It follows from Proposition 2.2.1 that $L^{p,\lambda}(B_\rho) \subset L^{p,\bar{\lambda}}(B_\rho)$ and $L^{q,\mu}(B_\rho) \subset L^{q,\bar{\mu}}(B_\rho)$. In particular, since $\varphi \in L^{p,\lambda}$ and $\psi \in L^{q,\mu}$, we get from the definition

$$\begin{aligned}
\|\varphi\|_{L^{p,\bar{\lambda}}(B_\rho)}^p &\leq \|\varphi\|_{L^{p,\lambda}(B_\rho)}^p \rho^{\lambda-\bar{\lambda}}, \\
\|\psi\|_{L^{q,\bar{\mu}}(B_\rho)}^q &\leq \|\psi\|_{L^{q,\mu}(B_\rho)}^q \rho^{\mu-\bar{\mu}}.
\end{aligned} \tag{3.39}$$

In fact, for any $B_s(y) \subset \mathbb{R}^n$, we have

$$\begin{aligned}
\frac{1}{s^{\bar{\lambda}}} \int_{B_\rho \cap B_s(y)} \varphi^p dx &= s^{\lambda-\bar{\lambda}} \frac{1}{s^\lambda} \int_{B_\rho \cap B_s(y)} \varphi^p dx \leq \rho^{\lambda-\bar{\lambda}} \frac{1}{s^\lambda} \int_{B_\rho \cap B_s(y)} \varphi^p dx \\
&\leq \|\varphi\|_{L^{p,\lambda}(B_\rho)}^p \rho^{\lambda-\bar{\lambda}}
\end{aligned}$$

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if $\rho \geq s$ and

$$\begin{aligned} \frac{1}{s^{\bar{\lambda}}} \int_{B_\rho \cap B_s(y)} \varphi^p dx &\leq \frac{1}{\rho^{\bar{\lambda}}} \int_{B_\rho \cap B_s(y)} \varphi^p dx \leq \rho^{\lambda - \bar{\lambda}} \frac{1}{\rho^{\bar{\lambda}}} \int_{B_\rho} \varphi^p dx \\ &\leq \|\varphi\|_{L^{p,\lambda}(B_\rho)}^p \rho^{\lambda - \bar{\lambda}} \end{aligned}$$

if $\rho \leq s$.

Take now a cut-off function $\eta \in C_0^\infty(B_{\rho/2})$ so that $0 \leq \eta \leq 1$, $\eta = 1$ on $B_{\rho/4}$ and $|D\eta| \leq C/\rho$. Since $\mathbb{R}^n \setminus \Omega$ is P -thick, it is also m -thick whence, according to (2.1),

$$\text{Cap}_m(B_{\rho/4} \setminus \Omega, B_{\rho/2}) \geq A_\Omega \text{Cap}_m(B_{\rho/4}, B_{\rho/2}) = C\rho^{n-m} \quad \forall \rho \leq r_0.$$

On the other hand,

$$B_{\rho/4} \setminus \Omega \subset \{x \in B_{\rho/4} : \bar{u}(x) = 0\}$$

and therefore

$$\text{Cap}_m(\{x \in B_{\rho/4} : \bar{u}(x) = 0\}, B_{\rho/2}) \geq \text{Cap}_m(B_{\rho/4} \setminus \Omega, B_{\rho/2}).$$

We have $\eta w^{-1} = M(\rho) + \bar{A}(\rho)$ on $\{x \in B_{\rho/4} : \bar{u}(x) = 0\}$ with

$$\bar{A}(\rho) = \rho + \|\varphi\|_{L^{p,\bar{\lambda}}(B_\rho)}^{\frac{1}{m-1}} + \|\psi\|_{L^{q,\bar{\mu}}(B_\rho)}^{\frac{1}{m}}$$

and thus

$$\int_{B_{\rho/2}} \left| D \left(\frac{\eta w^{-1}}{M(\rho) + \bar{A}(\rho)} \right) \right|^m dx \geq \text{Cap}_m(\{x \in B_{\rho/4} : \bar{u}(x) = 0\}, B_{\rho/2}).$$

Putting together all these inequalities, Lemma 3.4.1 gives

$$\begin{aligned} \rho^{n-m} &\leq C \text{Cap}_p(\{x \in B_{\rho/4} : \bar{u}(x) = 0\}, B_{\rho/2}) \\ &\leq C((M(\rho) + \bar{A}(\rho))^{-m} \int_{B_{\rho/2}} |D(\eta w^{-1})|^m \\ &\leq C(M(\rho) + \bar{A}(\rho))^{1-m} \left(M(\rho) - M\left(\frac{\rho}{2}\right) + \bar{A}(\rho) \right)^{m-1} \rho^{n-m}. \end{aligned}$$

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Thus, we find

$$M\left(\frac{\rho}{2}\right) \leq \frac{C-1}{C} (M(\rho) + \overline{A}(\rho))$$

for all $\rho \leq R$ where R depends on r_0 from (2.1), and it follows from Proposition 2.2.6 that

$$M(\rho) \leq C \left(\left(\frac{\rho}{R} \right)^{\alpha'} M(R) + \overline{A}(\rho^\tau R^{1-\tau}) \right)$$

for any $0 < \tau < 1$ and $\rho \leq R$ with an exponent $\alpha' > 0$. Since

$$\overline{A}(\rho^\tau R^{1-\tau}) \leq C(\rho^\tau R^{1-\tau})^{\alpha''}$$

with $\alpha'' = \min\left\{\frac{\lambda-\bar{\lambda}}{(m-1)p}, \frac{\mu-\bar{\mu}}{mq}, 1\right\}$ as it follows from (3.39), we have

$$M(\rho) \leq C\rho^\alpha$$

where $\alpha = \min\{\alpha', \tau\alpha''\}$.

Repeating the above procedure with $-u(x)$ instead of $u(x)$, we get finally

$$\sup_{B_\rho(x_0)} |u| \leq C\rho^\alpha \tag{3.40}$$

for all $x_0 \in \partial\Omega$ and all $\rho \in (0, R)$.

With Corollary 3.1.2 and (3.40) at hand, it is standard matter to get Hölder continuity up to the boundary as claimed in Theorem 3.1.3. For, we will distinguish between various cases for arbitrary two points $x, y \in \overline{\Omega}$.

Case 1: $\text{dist}(x, \partial\Omega) \geq R/2$ and $\text{dist}(y, \partial\Omega) \geq R/2$. Then

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq H$$

as it follows from Corollary 3.1.2.

Case 2: $0 < \text{dist}(x, \partial\Omega), \text{dist}(y, \partial\Omega) < R/2$. Let $\delta = \text{dist}(y, \partial\Omega)$. We have

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq \frac{\text{osc}_{B_{|x-y|}(y)} u}{|x - y|^\alpha},$$

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while

$$\operatorname{osc}_{B_{|x-y|}(y)} u \leq C|x-y|^\alpha \left(\delta^{-\alpha} \operatorname{osc}_{B_\delta(y)} u + 1 \right)$$

as consequence of [24, Theorem 4.1]. This way,

$$\frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C \left(\delta^{-\alpha} \operatorname{osc}_{B_\delta(y)} u + 1 \right)$$

for each $x \in B_\delta(y)$ with a suitable exponent $\alpha \in (0, 1)$. Pick now a point $y_0 \in \partial\Omega$ with the property $|y_0 - y| = \operatorname{dist}(y, \partial\Omega)$. Since $B_\delta(y) \subset B_{2\delta}(y_0)$, we have from (3.40) that

$$\operatorname{osc}_{B_\delta(y)} u \leq \operatorname{osc}_{B_{2\delta}(y_0)} u \leq 2 \sup_{B_{2\delta}(y_0)} |u| \leq C\delta^\alpha$$

whence

$$\frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq H$$

for all $x \in B_\delta(y)$ and all $y \in \Omega$ with $\operatorname{dist}(y, \partial\Omega) < R/2$. Further, if $|x-y| \geq \delta$ and $\operatorname{dist}(x, \partial\Omega) < R/2$, take a point $x_0 \in \partial\Omega$ with the property $|x_0 - x| = \operatorname{dist}(x, \partial\Omega)$. Since

$$|x - x_0| \leq |x - y_0| \leq |x - y| + |y - y_0| = |x - y| + \delta \leq 2|x - y|$$

and $u(x_0) = u(y_0) = 0$, we have

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(x_0)| + |u(x_0) - u(y_0)| + |u(y_0) - u(y)| \\ &\leq C(|x - x_0|^\alpha + |y - y_0|^\alpha) \\ &\leq C|x - y|^\alpha, \end{aligned}$$

whence

$$\frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq H$$

for all $x, y \in \Omega$ with $\operatorname{dist}(x, \partial\Omega), \operatorname{dist}(y, \partial\Omega) \in (0, R/2)$ and such that $|x - y| \geq \delta$.

Case 3: $\operatorname{dist}(x, \partial\Omega) \geq R/2$ and $0 < \operatorname{dist}(y, \partial\Omega) < R/2$. It suffices to take a point z lying on the segment with end x and y and such that $\operatorname{dist}(z, \partial\Omega) =$

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$R/2$ to get

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq \frac{|u(x) - u(z)|}{|x - z|^\alpha} + \frac{|u(z) - u(y)|}{|z - y|^\alpha}.$$

Thus, the desired estimate reduces to the cases already considered.

Case 4: $y \in \partial\Omega$. It follows from (3.40) that

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq H$$

for all $x \in \bar{\Omega}$ such that $|x - y| < R$, while

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq 2\|u\|_{L^\infty(\Omega)} R^{-\alpha} \leq H$$

if $|x - y| \geq R$, as consequence of (3.5).

It remains to take the smallest of the exponents α in the above considerations to complete the proof of Theorem 3.1.3. \square

3.4.2 Hölder continuity under natural structure conditions

Theorem 3.1.3 asserts global Hölder continuity of the weak solutions to the Dirichlet problem (1.1) under the same hypotheses which ensure global boundedness of the solutions. However, it happens very often that one already disposes of an *a priori* bound for $\|u\|_{L^\infty(\Omega)}$ as consequence, for example, of strong monotonicity of the principal part $\mathbf{a}(x, u, Du)$ with respect to Du , or sign condition on $u \cdot b(x, u, Du)$ (see e.g. [27, 28] and the references therein), etc. What is the natural question to arise in this situation is whether the *bounded* weak solutions to (1.1) remain globally Hölder continuous in Ω if the $|\xi|^{m(1-\frac{1}{\ell})}$ -growth of $b(x, z, \xi)$ in (3.1) is relaxed to $|\xi|^m$.

More precisely, let us weaken the *controlled* growth assumptions (3.1) to the *natural structure* conditions of Ladyzhenskaya and Ural'tseva. In other words, let $\varphi \in L^{p,\lambda}(\Omega)$ with $p > \frac{m}{m-1}$, $\lambda \in (0, n)$ and $(m-1)p + \lambda > n$; $\psi \in L^{q,\mu}(\Omega)$ with $q > \frac{mn}{mn+m-n}$, $\mu \in (0, n)$ and $mq + \mu > n$, and suppose there exist a non-decreasing function $\Lambda(t)$ and a non-increasing function $\gamma(t)$, both

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positive and continuous, such that

$$\begin{cases} |\mathbf{a}(x, z, \xi)| \leq \Lambda(|u|) (\varphi(x) + |\xi|^{m-1}), \\ |b(x, z, \xi)| \leq \Lambda(|u|) (\psi(x) + |\xi|^m) \end{cases} \quad (3.41)$$

and

$$\mathbf{a}(x, z, \xi) \cdot \xi \geq \gamma(|u|)|\xi|^m - \Lambda(|u|)\varphi(x)^{\frac{m}{m-1}} \quad (3.42)$$

for a.a. $x \in \Omega$ and all $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

Indeed, a *bounded weak solution* to (1.1) is a function $u \in L^\infty(\Omega) \cap W_0^{1,m}(\Omega)$ such that

$$\int_{\Omega} \mathbf{a}(x, u(x), Du(x)) \cdot Dv(x) \, dx + \int_{\Omega} b(x, u(x), Du(x))v(x) \, dx = 0$$

for each test function $v(x) \in L^\infty(\Omega) \cap W_0^{1,m}(\Omega)$.

It is worth noting that in the proof of Corollary 3.1.2 above, we reduced (3.1) and (3.3) just to (3.41) and (3.42), respectively. Further, it is easy to check that the result of Lemma 3.4.1 remains valid for *bounded* weak solutions to (1.1) if (3.41) and (3.42) are required instead of (3.1) and (3.3). This way, we have

Theorem 3.4.2. *Under the hypotheses (2.1), (3.41) and (3.42), each bounded weak solution of the Dirichlet problem (1.1) is Hölder continuous in $\overline{\Omega}$ with Hölder exponent and constant depending on the data of (1.1) and on $\|u\|_{L^\infty(\Omega)}$.*

Chapter 4

Morrey regularity

4.1 Hypotheses and Main Results

In this chapter, we consider the Dirichlet problem

$$\begin{cases} \operatorname{div}(\mathbf{a}(x, u, Du) + \mathbf{b}(x, u)) = c(x, u, Du) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

given in (1.3). The nonlinearities considered are given by the Carathéodory maps $\mathbf{a}: \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\mathbf{b}: \Omega \times \mathbb{R} \rightarrow \mathbb{R}^n$ and $c: \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, where $\mathbf{a}(x, z, \xi) = (a^1(x, z, \xi), \dots, a^n(x, z, \xi))$ and $\mathbf{b}(x, z) = (b^1(x, z), \dots, b^n(x, z))$. We suppose moreover that $\mathbf{a}(x, z, \xi)$ is differentiable with respect to ξ and $D_\xi \mathbf{a}$ is a Carathéodory map.

Throughout this chapter the following structure and regularity conditions on the data will be assumed:

- *Uniform ellipticity:* There exists a constant $\gamma > 0$ such that

$$\begin{cases} \gamma |\xi|^{m-2} |\eta|^2 \leq \langle D_\xi \mathbf{a}(x, z, \xi) \eta, \eta \rangle, \\ |\mathbf{a}(x, z, \xi)| + |\xi| |D_\xi \mathbf{a}(x, z, \xi)| \leq \gamma^{-1} |\xi|^{m-1} \end{cases} \quad (4.1)$$

for a.a. $x \in \Omega$ and $\forall (z, \xi) \in \mathbb{R} \times \mathbb{R}^n$, $\forall \eta \in \mathbb{R}^n$.

- *Controlled growth conditions:* There exist a constant $\Lambda > 0$ and non-negative functions $\varphi \in L^{p, \lambda}(\Omega)$ with $p > \frac{m}{m-1}$, $\lambda \in (0, n)$ and $(m-1)p + \lambda > n$,

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and $\psi \in L^{q,\mu}(\Omega)$ with $q > \frac{mn}{mn+m-n}$, $\mu \in (0, n)$ and $mq + \mu > n$, such that

$$\begin{cases} |\mathbf{b}(x, z)| \leq \Lambda \left(\varphi(x) + |z|^{\frac{\ell(m-1)}{m}} \right), \\ |c(x, z, \xi)| \leq \Lambda \left(\psi(x) + |z|^{\ell-1} + |\xi|^{\frac{m(\ell-1)}{\ell}} \right) \end{cases} \quad (4.2)$$

for a.a. $x \in \Omega$ and all $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$. Here, ℓ is the Sobolev conjugate of m and is given by (3.2).

• *Local uniform continuity:* For each $M > 0$ there is a non-decreasing function $\sigma_M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\lim_{t \rightarrow 0+} \sigma_M(t) = 0$ and such that

$$|\mathbf{a}(x, z_1, \xi) - \mathbf{a}(x, z_2, \xi)| \leq \sigma_M(|z_1 - z_2|)|\xi|^{m-1} \quad (4.3)$$

for a.a. $x \in \Omega$, $\forall z_1, z_2 \in [-M, M]$ and $\forall \xi \in \mathbb{R}^n$.

• *(δ, R) -vanishing property:* For each constant $M > 0$ there exist $R_M > 0$ and $\delta_M > 0$ such that

$$\sup_{z \in [-M, M]} \sup_{0 < \rho \leq R_M} \sup_{y \in \mathbb{R}^n} \int_{B_\rho(y)} \Theta(\mathbf{a}; B_\rho(y))(x, z) dx \leq \delta_M. \quad (4.4)$$

Here the function Θ is defined by

$$\Theta(\mathbf{a}; B_\rho(y))(x, z) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|\mathbf{a}(x, z, \xi) - \bar{\mathbf{a}}_{B_\rho(y)}(z, \xi)|}{|\xi|^{m-1}}, \quad (4.5)$$

where $\bar{\mathbf{a}}_{B_\rho(y)}(z, \xi)$ is the integral average of $\mathbf{a}(x, z, \xi)$ in the variables x for the fixed $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$, that is,

$$\bar{\mathbf{a}}_{B_\rho(y)}(z, \xi) = \int_{B_\rho(y)} \mathbf{a}(x, z, \xi) dx.$$

Recall that a function $u \in W_0^{1,m}(\Omega)$ is called *weak solution* to the Dirichlet problem (1.1) if

$$\int_{\Omega} (\mathbf{a}(x, u(x), Du(x)) + \mathbf{b}(x, u(x))) \cdot Dv(x) dx = \int_{\Omega} c(x, u(x), Du(x))v(x) dx \quad (4.6)$$

for each test function $v \in W_0^{1,m}(\Omega)$. It is worth noting that the convergence of the integrals involved in (4.4) for all admissible u and v is ensured by (4.1)–

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(4.2) under the *sole* assumptions $p \geq \frac{m}{m-1}$ and $q \geq \frac{mn}{mn+m-n}$ when $m < n$, $q > 1$ if $m = n$.

In what follows, given any two exponents $s > 1$ and $\theta \in (0, n)$, we set

$$s_\theta^* = \begin{cases} \frac{(n-\theta)s}{n-\theta-s} & \text{if } s + \theta < n, \\ \text{arbitrary large number} & \text{if } s + \theta \geq n \end{cases} \quad (4.7)$$

for the *Sobolev-Morrey* conjugate of s .

Throughout this chapter the omnibus phrase “*known quantities*” means that a given constant depends on the data in hypotheses (4.1)–(4.4), which include $n, m, \ell, \gamma, \Lambda, p, \lambda, q, \mu, \|\varphi\|_{L^{p,\lambda}(\Omega)}, \|\psi\|_{L^{q,\mu}(\Omega)}, \text{diam } \Omega, \delta, R, \sigma_M, R_M$ and δ_M . We will denote by C a generic constant, depending on known quantities, which may vary within the same formula.

The main result of this chapter is the following.

Theorem 4.1.1. *Suppose (4.1)–(4.4), and let $u \in W_0^{1,m}(\Omega)$ be a weak solution to the Dirichlet problem (1.3). Then there exists a small $\delta_0 > 0$ such that if Ω is (δ, R) -Reifenberg flat and $\mathbf{a}(x, z, \xi)$ is (δ, R) -vanishing in the sense of (4.4) with $\delta < \delta_0$, then*

$$u \in L^\infty(\Omega) \cap W^{1,(m-1)r}(\Omega) \quad \text{with } r = \min\{p, q_\mu^*\}.$$

Moreover, the gradient Du belongs to an appropriate Morrey space. Precisely,

- (1) If $q + \mu \geq n$ then $Du \in L^{(m-1)p,\lambda}$.
- (2) If $q + \mu < n$ then $Du \in L^{(m-1)r,\nu}$ with $r = \min\{p, q_\mu^*\}$ and $\nu = \min\left\{\frac{r(\mu-n)}{q_\mu^*} + n, \frac{r(\lambda-n)}{p} + n\right\}$.

An immediate consequence of Theorem 4.1.1, Proposition 2.2.8, $mq + \mu > n$ and $(m-1)p + \lambda > n$, is the following global Hölder continuity of the weak solution to (1.3).

Corollary 4.1.2. *Under the assumptions of Theorem 4.1.1, a $W_0^{1,m}(\Omega)$ -weak solution to the problem (1.3) is Hölder continuous in $\bar{\Omega}$.*

More precisely,

$$u \in C^{0,\min\left\{1-\frac{n-\lambda}{(m-1)p}, 1-\frac{n-\mu-q}{(m-1)q}\right\}}(\Omega) \equiv C^{0,\min\left\{\frac{(m-1)p+\lambda-n}{(m-1)p}, \frac{mq+\mu-n}{(m-1)q}\right\}}(\Omega).$$

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In particular, when $m > n$, one can easily check that

$$\min \left\{ 1 - \frac{n - \lambda}{(m - 1)p}, 1 - \frac{n - \mu - q}{(m - 1)q} \right\} > 1 - \frac{n}{m}.$$

As a consequence, we achieve a better Hölder regularity than that from the Morrey embedding inequality.

4.2 Nonlinear elliptic equations

Consider the following Dirichlet problem

$$\begin{cases} \operatorname{div} (\mathbf{A}(x, Du(x))) = \operatorname{div} (|\mathbf{F}(x)|^{m-2} \mathbf{F}(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.8)$$

where $\mathbf{F} = (f^1, \dots, f^n) \in L^m(\Omega, \mathbb{R}^n)$ is a given vector-valued function and the nonlinearity $\mathbf{A}(x, \xi)$ together with $D_\xi \mathbf{A}(x, \xi)$ are Carathéodory maps satisfying the following condition:

$$\begin{cases} \gamma |\xi|^{m-2} |\eta|^2 \leq \langle D_\xi \mathbf{A}(x, \xi) \eta, \eta \rangle, \\ |\mathbf{A}(x, \xi)| + |\xi| |D_\xi \mathbf{A}(x, \xi)| \leq \gamma^{-1} |\xi|^{m-1}. \end{cases} \quad (4.9)$$

According to *Minty-Browder method*, (4.8) possesses a unique weak solution $u \in W_0^{1,m}(\Omega)$ and

$$\|Du\|_{L^m(\Omega)} \leq c \|\mathbf{F}\|_{L^m(\Omega)}$$

with a constant c depending only on γ , n , m and Ω .

Definition 4.2.1. *We say that the vector field \mathbf{A} is (δ, R) -vanishing if*

$$\sup_{0 < \rho \leq R} \sup_{y \in \mathbb{R}^n} \int_{B_\rho(y)} \Theta(\mathbf{A}; B_\rho(y))(x) dx \leq \delta. \quad (4.10)$$

As consequence of the weighted $L^p(\Omega)$ -theory, we have the following result regarding gradient Morrey regularity of the weak solution to (4.8).

Lemma 4.2.2. (see [6]) *For each $s \in (m, \infty)$ and each $\theta \in [0, n)$, there exist a small positive constant δ and a constant C , depending on γ , Λ , n , s , θ and Ω , such that if \mathbf{A} is (δ, R) -vanishing, Ω is (δ, R) -Reifenberg flat and*

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$\mathbf{F} \in L^{s,\theta}(\Omega, \mathbb{R}^n)$, then the unique weak solution $u \in W_0^{1,m}(\Omega)$ to the problem (4.8) satisfies $Du \in L^{s,\theta}(\Omega)$ and

$$\|Du\|_{L^{s,\theta}(\Omega)} \leq C\|\mathbf{F}\|_{L^{s,\theta}(\Omega, \mathbb{R}^n)}.$$

4.3 (δ, R) -vanishing properties of superposition operators

Lemma 4.3.1. *Under the assumptions (4.3) and (4.4), for each $u \in C^0(\overline{\Omega})$ the composition $\mathbf{A}(x, \xi) := \mathbf{a}(x, u(x), \xi)$ is (δ, R) -vanishing. Precisely,*

$$\sup_{0 < \rho \leq R} \sup_{y \in \mathbb{R}^n} \int_{B_\rho(y)} \Theta(\mathbf{A}; B_\rho(y))(x) dx \leq \delta, \quad (4.11)$$

where

$$\begin{aligned} \Theta(\mathbf{A}; B_\rho(y))(x) &= \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|\mathbf{A}(x, \xi) - \overline{\mathbf{A}}_{B_\rho(y)}(\xi)|}{|\xi|^{m-1}}, \\ \delta &= \delta_M + 2\sigma_M(\omega_u(R_M)), \\ R &= R_M, \end{aligned}$$

$M = \|u\|_{L^\infty(\Omega)}$ and $\omega_u(\cdot)$ is the modulus of continuity of the function u .

Proof. Set $M = \|u\|_{L^\infty(\Omega)}$ and $R = R_M$ as given by (4.4). Let $x, y \in \mathbb{R}^n$, $0 < \rho \leq R$, $|x - y| < \rho$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. In view of triangle inequality, we have

$$\begin{aligned} |\mathbf{A}(x, \xi) - \overline{\mathbf{A}}_{B_\rho(y)}(\xi)| &\leq |\mathbf{a}(x, u(x), \xi) - \mathbf{a}(x, u(y), \xi)| \\ &\quad + |\overline{\mathbf{a}}_{B_\rho(y)}(u(y), \xi) - \overline{\mathbf{A}}_{B_\rho(y)}(\xi)| \\ &\quad + |\mathbf{a}(x, u(y), \xi) - \overline{\mathbf{a}}_{B_\rho(y)}(u(y), \xi)| \\ &=: J_1 + J_2 + J_3. \end{aligned}$$

Since $|u| \leq M$ in Ω , local uniform continuity (4.3) gives

$$J_1 = |\mathbf{a}(x, u(x), \xi) - \mathbf{a}(x, u(y), \xi)| \leq \sigma_M(|u(x) - u(y)|)|\xi|^{m-1}.$$

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Employing the modulus of continuity of u , we get

$$J_1 \leq \sigma_M(|u(x) - u(y)|)|\xi|^{m-1} \leq \sigma_M(\omega_u(R))|\xi|^{m-1}$$

In a similar manner one has

$$\begin{aligned} J_2 &\leq \int_{B_\rho(y)} |\mathbf{a}(\tilde{x}, u(y), \xi) - \mathbf{a}(\tilde{x}, u(\tilde{x}), \xi)| d\tilde{x} \\ &\leq \int_{B_\rho(y)} \sigma_M(|u(y) - u(\tilde{x})|)|\xi|^{m-1} d\tilde{x} \\ &\leq \sigma_M(\omega_u(R))|\xi|^{m-1}, \end{aligned}$$

while

$$J_3 \leq \Theta(\mathbf{a}; B_\rho(y))(x, u(y))|\xi|^{m-1}$$

with Θ given in (4.5).

Therefore we have

$$\Theta(\mathbf{A}; B_\rho(y))(x) \leq \Theta(\mathbf{a}; B_\rho(y))(x, u(y)) + 2\sigma_M(\omega_u(R)),$$

$$\begin{aligned} \int_{B_\rho(y)} \Theta(\mathbf{A}; B_\rho(y))(x) dx &\leq \int_{B_\rho(y)} \Theta(\mathbf{a}; B_\rho(y))(x, u(y)) dx + 2\sigma_M(\omega_u(R)) \\ &\leq \delta_M + 2\sigma_M(\omega_u(R)). \end{aligned}$$

□

4.4 Proof of the main results

Without loss of generality, we assume that the solution u and the data φ and ψ are extended as zero outside Ω .

Step 1: For, fix the solution $u \in W_0^{1,m}(\Omega) \cap L^\infty(\Omega)$ into nonlinear terms of (1.3) and define

$$\mathbf{A}(x, \xi) := \mathbf{a}(x, u(x), \xi), \quad \mathbf{B}(x) := -\mathbf{b}(x, u(x)), \quad f(x) := c(x, u(x), Du(x)). \quad (4.12)$$

Set $\Gamma(x - y)$ for the normalized fundamental solution of the Laplace operator

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and let

$$\mathcal{N}f(x) := \int_{\Omega} \Gamma(x-y)f(y) \, dy$$

with the Newtonian potential with density f . Since $f \in L^{\min\{q, \frac{\ell}{\ell-1}\}}(\Omega)$ by (4.2), the Newtonian potential is well-defined and it is well-known that $\mathcal{N}f \in W^{2, \min\{q, \frac{\ell}{\ell-1}\}}(\Omega)$ and $\Delta(\mathcal{N}f(x)) = f(x)$ for a.a. $x \in \Omega$. Moreover, defining

$$\bar{\mathbf{F}} := D(\mathcal{N}f(x)) = \int_{\Omega} D_x \Gamma(x-y)f(y) \, dy = C(n) \int_{\Omega} \frac{x-y}{|x-y|^n} f(y) \, dy,$$

it is clear that $\operatorname{div} \bar{\mathbf{F}}(x) = f(x)$ for a.a. $x \in \Omega$.

Setting

$$\mathbf{F}(x) := \begin{cases} |\mathbf{B}(x) + \bar{\mathbf{F}}(x)|^{\frac{2-m}{m-1}} (\mathbf{B}(x) + \bar{\mathbf{F}}(x)) & \text{if } |\mathbf{B}(x) + \bar{\mathbf{F}}(x)| > 0, \\ 0 & \text{if } |\mathbf{B}(x) + \bar{\mathbf{F}}(x)| = 0, \end{cases}$$

we have

$$|\mathbf{F}|^{m-2} \mathbf{F} = \mathbf{B} + \bar{\mathbf{F}}.$$

Therefore, (1.3) can be rewritten into

$$\begin{cases} \operatorname{div} (\mathbf{A}(x, Du(x))) = \operatorname{div} (|\mathbf{F}(x)|^{m-2} \mathbf{F}(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.13)$$

with the term $\mathbf{A}(x, \xi)$ of (δ, R) -vanishing which satisfies (4.9) as consequence of (4.1), Lemma 4.3.1 and Theorem 3.1.3. Further on, it follows from (4.2)

$$\begin{aligned} |\bar{\mathbf{F}}(x)| &\leq C \int_{\Omega} \frac{|f(y)|}{|x-y|^{n-1}} \, dy \\ &\leq C \int_{\Omega} \frac{1}{|x-y|^{n-1}} \left(1 + \psi(y) + |Du(y)|^{\frac{m(\ell-1)}{\ell}} \right) \, dy \\ &= C \left(I_1(1)(x) + I_1(\psi)(x) + I_1 \left(|Du|^{\frac{m(\ell-1)}{\ell}} \right)(x) \right) \end{aligned}$$

and therefore we have

$$|\mathbf{F}(x)|^{m-1} \leq C \left(I_1 \left(|Du|^{\frac{m(\ell-1)}{\ell}} \right)(x) + I_1(\psi)(x) + \varphi(x) + 1 \right) \quad (4.14)$$

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with C depending on known quantities and on $\|Du\|_{L^m(\Omega)}$ in addition.

Step 2: In this step, we will show that

$$|Du|^{m-1} \in L^{\min\{q_\mu^*, p\}}(\Omega).$$

Employing *the higher integrability of the gradient Du* of Lemma 3.2.1, we may assume

$$|Du|^{m-1} \in L^s(\Omega)$$

for some $s \in [\frac{m_0}{m-1}, \infty)$. This is equivalent to $|Du|^{\frac{m(\ell-1)}{\ell}} \in L^{\frac{(m-1)\ell}{m(\ell-1)}s}(\Omega)$ and Proposition 2.2.7 yields

$$I_1 \left(|Du|^{\frac{m(\ell-1)}{\ell}} \right) \in L^{\left(\frac{(m-1)\ell}{m(\ell-1)}s \right)^*}(\Omega) \equiv L^{\left(\frac{(m-1)\ell}{m(\ell-1)}s \right)^*, 0}(\Omega). \quad (4.15)$$

When $\frac{(m-1)\ell}{m(\ell-1)}s < n$, we have

$$\left(\frac{(m-1)\ell}{m(\ell-1)}s \right)^* = \frac{(m-1)\ell n s}{mn(\ell-1) - (m-1)\ell s} \geq \frac{(m-1)\ell n s}{mn(\ell-1) - m_0\ell}$$

because of $(m-1)s \geq m_0$. Defining

$$\tau_0 := \frac{(m-1)\ell n}{mn(\ell-1) - m_0\ell},$$

it follows

$$\tau_0 = \frac{(m-1)\ell n}{mn(\ell-1) - m_0\ell} = \frac{(m-1)\ell n}{(m-1)\ell n + (n-m)\ell - nm - (m_0-m)\ell} > 1$$

since $m_0 > m$ and $(n-m)\ell \leq mn$. Therefore, we have

$$\left(\frac{(m-1)\ell}{m(\ell-1)}s \right)^* \geq \tau_0 s > s. \quad (4.16)$$

Further on, (4.2) and Proposition 2.2.7 yield

$$I_1 \psi \in L^{q_\mu^*, \mu}(\Omega) \quad (4.17)$$

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and therefore we have

$$|\mathbf{F}|^{m-1} \in L^{\min\{\tau_0 s, q_\mu^*, p\}}(\Omega)$$

as consequence of (4.2) and (4.14)–(4.16). Noting that

$$\min\{\tau_0 s, q_\mu^*, p\} > \frac{m}{m-1},$$

Lemma 4.2.2 implies

$$|Du|^{m-1} \in L^{\min\{\tau_0 s, q_\mu^*, p\}}(\Omega).$$

To proceed further, we take $s = s_k = \tau_0^k \frac{m_0}{m-1}$ for $k = 0, 1, 2, \dots$. Keeping in mind Lemma 3.2.1, we iterate the above procedure until $s_k \geq \min\left\{\frac{(\ell-1)mn}{\ell(m-1)}, q_\mu^*, p\right\}$.

This way, we get

$$|Du|^{m-1} \in L^{\min\left\{\frac{(\ell-1)mn}{\ell(m-1)}, q_\mu^*, p\right\}}(\Omega).$$

In particular, if $\frac{(\ell-1)mn}{\ell(m-1)} \geq \min\{q_\mu^*, p\}$, then $|Du|^{m-1} \in L^{\min\{q_\mu^*, p\}}(\Omega)$ and we obtain the claim of Step 2.

If instead $\frac{(\ell-1)mn}{\ell(m-1)} < \min\{q_\mu^*, p\}$ then $|Du|^{m-1} \in L^{\frac{(\ell-1)mn}{\ell(m-1)}}(\Omega)$ and we take $s = \frac{(\ell-1)mn}{\ell(m-1)}$ that gives $\frac{(m-1)\ell}{m(\ell-1)}s = n$. Let us consider therefore the case $\frac{(m-1)\ell}{m(\ell-1)}s \geq n$. Choosing the Sobolev conjugate as

$$\left(\frac{(m-1)\ell}{m(\ell-1)}s\right)^* = \min\{q_\mu^*, p\},$$

we get

$$|\mathbf{F}|^{m-1} \in L^{\min\{q_\mu^*, p\}}(\Omega)$$

as consequence of (4.2), (4.15) and (4.17), and thus we have

$$|Du|^{m-1} \in L^{\min\{q_\mu^*, p\}}(\Omega).$$

Setting hereafter $r := \min\{q_\mu^*, p\}$, we have $Du \in L^{(m-1)r}(\Omega)$ that is $u \in W^{1, (m-1)r}(\Omega)$.

Step 3: To obtain the Morrey regularity of Du as claimed in Theo-

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rem 4.1.1, we assume that $|Du|^{m-1} \in L^{r,\theta}(\Omega)$ for some $\theta \in [0, n)$. This is equivalent to $|Du|^{\frac{m(\ell-1)}{\ell}} \in L^{\frac{(m-1)\ell}{m(\ell-1)}r,\theta}(\Omega)$ and Proposition 2.2.7 yields

$$I_1 \left(|Du|^{\frac{m(\ell-1)}{\ell}} \right) \in L^{\left(\frac{(m-1)\ell}{m(\ell-1)}r \right)^*, \theta}(\Omega).$$

If $\frac{(m-1)\ell}{m(\ell-1)}r < n - \theta$, we have

$$\left(\frac{(m-1)\ell}{m(\ell-1)}r \right)^*_\theta = \frac{(n-\theta)(m-1)\ell r}{(n-\theta)m(\ell-1) - (m-1)\ell r}.$$

and Proposition 2.2.1 gives the inclusion

$$L^{\left(\frac{(m-1)\ell}{m(\ell-1)}r \right)^*, \theta}(\Omega) \subset L^{r, n+r-(n-\theta)\frac{m(\ell-1)}{(m-1)\ell}}(\Omega).$$

Further on, set

$$\tau_1 := \frac{n(m-\ell) + r\ell(m-1)}{\ell(m-1)}, \quad \tau_2 := \frac{m(\ell-1)}{(m-1)\ell}.$$

Keeping in mind $r > \frac{m}{m-1}$ and $(n-m)\ell \leq mn$, we get

$$\begin{aligned} \tau_1 &= \frac{n(m-\ell) + r\ell(m-1)}{\ell(m-1)} > \frac{n(m-\ell) + \ell m}{\ell(m-1)} \\ &= \frac{mn - \ell(n-m)}{\ell(m-1)} \geq 0. \end{aligned}$$

On the other hand, it is clear

$$\tau_2 = \frac{m(\ell-1)}{(m-1)\ell} > 1$$

as consequence of $\ell > m$, and therefore straightforward calculations show

$$\begin{aligned} n + r - (n-\theta)\frac{m(\ell-1)}{(m-1)\ell} &= \frac{n(m-\ell) + r\ell(m-1)}{\ell(m-1)} + \theta\frac{m(\ell-1)}{(m-1)\ell} \\ &= \tau_1 + \tau_2\theta > \theta. \end{aligned}$$

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Thus, we obtain

$$I_1 \left(|Du|^{\frac{m(\ell-1)}{\ell}} \right) \in L^{r, \tau_1 + \theta \tau_2}(\Omega).$$

In a similar manner of step 2, one has

$$|\mathbf{F}|^{m-1} \in L^{r, \min \left\{ \tau_1 + \theta \tau_2, \frac{\mu-n}{q_\mu^*} r + n, \frac{\lambda-n}{p} r + n \right\}}(\Omega)$$

whence applying the Lemma 4.2.2 again, we have

$$|Du|^{m-1} \in L^{r, \min \left\{ \tau_1 + \theta \tau_2, \frac{\mu-n}{q_\mu^*} r + n, \frac{\lambda-n}{p} r + n \right\}}(\Omega).$$

To proceed further, we first take $\theta = 0$, then we have

$$|Du|^{m-1} \in L^{r, \min \left\{ \tau_1, \frac{\mu-n}{q_\mu^*} r + n, \frac{\lambda-n}{p} r + n \right\}}(\Omega).$$

If $\tau_1 \geq \min \left\{ \frac{\mu-n}{q_\mu^*} r + n, \frac{\lambda-n}{p} r + n \right\}$ then we are done. Otherwise, $|Du|^{m-1} \in L^{r, \tau_1}(\Omega)$ and we repeat the above procedure with $\theta = \tau_1$, obtaining

$$|Du|^{m-1} \in L^{r, \min \left\{ \tau_1 + \tau_1 \tau_2, \frac{\mu-n}{q_\mu^*} r + n, \frac{\lambda-n}{p} r + n \right\}}(\Omega) \subset L^{r, \min \left\{ \tau_1 \tau_2, \frac{\mu-n}{q_\mu^*} r + n, \frac{\lambda-n}{p} r + n \right\}}(\Omega).$$

At this point we let $\theta = \theta_k = \tau_1 \tau_2^k$ for $k = 1, 2, \dots$ and iterate the above procedure. This gives

$$|Du|^{m-1} \in L^{r, \min \left\{ \tau_1 + \tau_1 \tau_2^k, \frac{\mu-n}{q_\mu^*} r + n, \frac{\lambda-n}{p} r + n \right\}}(\Omega) \subset L^{r, \min \left\{ \tau_1 \tau_2^k, \frac{\mu-n}{q_\mu^*} r + n, \frac{\lambda-n}{p} r + n \right\}}(\Omega).$$

We choose now k so large to have

$$\tau_1 \tau_2^k \geq \min \left\{ \frac{\mu-n}{q_\mu^*} r + n, \frac{\lambda-n}{p} r + n, n - \frac{(m-1)\ell}{m(\ell-1)} r \right\},$$

getting this way

$$|Du|^{m-1} \in L^{r, \min \left\{ \frac{\mu-n}{q_\mu^*} r + n, \frac{\lambda-n}{p} r + n, n - \frac{(m-1)\ell}{m(\ell-1)} r \right\}}(\Omega).$$

CHAPTER 4. MORREY REGULARITY

If

$$n - \frac{(m-1)\ell}{m(\ell-1)}r \geq \min \left\{ \frac{\mu-n}{q_\mu^*}r + n, \frac{\lambda-n}{p}r + n \right\}$$

then

$$|Du|^{m-1} \in L^{r, \min \left\{ \frac{\mu-n}{q_\mu^*}r + n, \frac{\lambda-n}{p}r + n \right\}}(\Omega)$$

and this completes the proof of Theorem 4.1.1.

If instead

$$n - \frac{(m-1)\ell}{m(\ell-1)}r < \min \left\{ \frac{\mu-n}{q_\mu^*}r + n, \frac{\lambda-n}{p}r + n \right\}$$

then

$$|Du|^{m-1} \in L^{r, n - \frac{(m-1)\ell}{m(\ell-1)}r}(\Omega)$$

and we take $\theta = n - \frac{(m-1)\ell}{m(\ell-1)}r$ that gives

$$\frac{(m-1)\ell}{m(\ell-1)}r = n - \theta.$$

So we consider now the case

$$\frac{(m-1)\ell}{m(\ell-1)}r \geq n - \theta$$

when

$$|Du|^{m-1} \in L^{r, \theta}(\Omega) \subset L^{r, n - \frac{(m-1)\ell}{m(\ell-1)}r}(\Omega).$$

We take the Sobolev-Morrey conjugate as

$$\left(\frac{(m-1)\ell}{m(\ell-1)}r \right)^*_{n - \frac{(m-1)\ell}{m(\ell-1)}r} = \frac{nr - \frac{(m-1)\ell}{m(\ell-1)}r^2}{n - \min \left\{ \frac{\mu-n}{q_\mu^*}r + n, \frac{\lambda-n}{p}r + n \right\}}$$

and then

$$|\mathbf{F}|^{m-1} \in L^{\frac{nr - \frac{(m-1)\ell}{m(\ell-1)}r^2}{n - \min \left\{ \frac{\mu-n}{q_\mu^*}r + n, \frac{\lambda-n}{p}r + n \right\}}, n - \frac{(m-1)\ell}{m(\ell-1)}r}(\Omega) \subset L^{r, \min \left\{ \frac{\mu-n}{q_\mu^*}r + n, \frac{\lambda-n}{p}r + n \right\}}(\Omega).$$

CHAPTER 4. MORREY REGULARITY

Applying Lemma 4.2.2 once again, we have

$$|Du|^{m-1} \in L^{r, \min\left\{\frac{\mu-n}{q_\mu^*}r+n, \frac{\lambda-n}{p}r+n\right\}}(\Omega)$$

which gives the claim of Theorem 4.1.1 also in this case. \square

Bibliography

- [1] D.R. Adams, Traces of potentials arising from translation invariant operators, *Ann. Scuola Norm. Sup. Pisa (3)* **25** (1971), 203–217.
- [2] D.R. Adams, A note on Riesz potentials, *Duke Math. J.* **42** (1975), no. 4, 765–778.
- [3] S.-S. Byun and D.K. Palagachev, Boundedness of the weak solutions to quasilinear elliptic equations with Morrey data, *Indiana Univ. Math. J.* **62** (2013), no. 5, 1565–1585.
- [4] S.-S. Byun and D.K. Palagachev, Morrey regularity of solutions to quasilinear elliptic equations over Reifenberg flat domains, *Calc. Var. Partial Differential Equations* **49** (2014), no. 1-2, 37–76.
- [5] S.-S. Byun and D.K. Palagachev, Weighted L^p -estimates for elliptic equations with measurable coefficients in nonsmooth domains, *Potential Anal.* **41** (2014), no. 1, 51–79.
- [6] S.-S. Byun, S. Ryu, Global weighted estimates for the gradient of solutions to nonlinear elliptic equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **30** (2013), no. 2, 291–313.
- [7] S.-S. Byun, L. Wang, Elliptic equations with BMO coefficients in Reifenberg domains, *Comm. Pure Appl. Math.* **57** (2004), no. 10, 1283–1310.
- [8] S.-S. Byun, L. Wang, Elliptic equations with BMO nonlinearity in Reifenberg domains, *Adv. Math.* **219** (2008), 1–24.
- [9] S. Campanato, *Sistemi ellittici in forma divergenza. Regolarità all'interno*, Pubblicazioni della Classe di Scienze: Quaderni, Scuola Norm. Sup., Pisa (1980).

BIBLIOGRAPHY

- [10] G. David, T. De Pauw and T. Toro, A generalization of Reifenberg's theorem in \mathbb{R}^3 , *Geom. Funct. Anal.* **18** (2008), no. 4, 1168–1235.
- [11] E. De Giorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)* (1957), 25–43.
- [12] R. Gariepy and W.P. Ziemer, A regularity condition at the boundary for solutions of quasilinear elliptic equations, *Arch. Rational Mech. Anal.* **67** (1977), no. 1, 25–39.
- [13] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Annals of Mathematics Studies, 105, Princeton University Press, Princeton, NJ, 1983.
- [14] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften, 224, Springer-Verlag, Berlin, 1983.
- [15] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, Oxford University Press, New York, 1993.
- [16] Ph. Hartman and G. Stampacchia, On some non-linear elliptic differential-functional equations, *Acta Math.* **115** (1966), 271–310.
- [17] T. Kilpeläinen, P. Koskela, Global integrability of the gradients of solutions to partial differential equations, *Nonlinear Anal.* **23** (1994), no. 7, 899–909.
- [18] O.A. Ladyzhenskaya and N.N. Ural'tseva, Quasilinear elliptic equations and variational problems in several independent variables, *Russian Math. Surveys* **16** (1961), no. 1, 17–91.
- [19] O.A. Ladyzhenskaya and N.N. Ural'tseva, On the Hölder continuity of the solutions and the derivatives of linear and quasi-linear equations of elliptic and parabolic types, *Trudy Mat. Inst. Steklov* **73** (1964), 172–220 (in Russian).
- [20] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Linear and Quasilinear Equations of Elliptic Type*, 2nd Edition revised, Nauka, Moscow, 1973, (in Russian).

BIBLIOGRAPHY

- [21] O.A. Ladyzhenskaya, N.N. Ural'tseva, On the smoothness of weak solutions of quasilinear equations in several variables and of variational problems, *Commun. Pure Appl. Math.* **14** (1961), 481–495.
- [22] J.L. Lewis, Uniformly fat sets, *Trans. Amer. Math. Soc.* **308** (1988), no. 1, 177–196.
- [23] H. Lewy and G. Stampacchia, On the smoothness of superharmonics which solve a minimum problem *J. Analyse Math.*, **23** (1970), 227–236.
- [24] G.M. Lieberman, Sharp forms of estimates for subsolutions and supersolutions of quasilinear elliptic equations involving measures, *Comm. Partial Differential Equations* **18** (1993), no. 7-8, 1191–1212.
- [25] T. Mengesha and N.C. Phuc, Weighted and regularity estimates for nonlinear equations on Reifenberg flat domains, *J. Differential Equations* **250** (2011), no. 5 2485–2507.
- [26] C.B. Morrey Jr., Second order elliptic equations in several variables and Hölder continuity, *Math. Z.* **72** (1959/1960), 146–164.
- [27] D.K. Palagachev, Global Hölder continuity of weak solutions to quasilinear divergence form elliptic equations, *J. Math. Anal. Appl.* **359** (2009), no. 1, 159–167.
- [28] D.K. Palagachev, Discontinuous superlinear elliptic equations of divergence form, *NoDEA Nonlinear Differential Equations Appl.* **16** (2009), no. 6, 811–822.
- [29] D.K. Palagachev, L. Recke, L.G. Softova, Applications of the differential calculus to nonlinear elliptic operators with discontinuous coefficients, *Math. Ann.* **336** (2006), no. 3, 617–637.
- [30] D.K. Palagachev and L.G. Softova, The Calderón–Zygmund property for quasilinear divergence form equations over Reifenberg flat domains, *Nonlinear Anal.* **74** (2011), no. 5, 1721–1730.
- [31] L.C. Piccinini, Inclusioni tra spazi di Morrey, *Boll. Un. Mat. Ital. (4)* **2** (1969), 95–99.

BIBLIOGRAPHY

- [32] J.-M. Rakotoson, Equivalence between the growth of $\int_{B(x,r)} |\nabla u|^p dy$ and T in the equation $P[u] = T$, *J. Differential Equations* **86** (1990), no. 1, 102–122.
- [33] J.-M. Rakotoson and W.P. Ziemer, Local behavior of solutions of quasi-linear elliptic equations with general structure, *Trans. Amer. Math. Soc.* **319** (1990), no. 2, 747–764.
- [34] E.R. Reifenberg, Solution of the Plateau problem for m -dimensional surfaces of varying topological type, *Acta Math.* **104** (1960), no. 1-2, 1–92.
- [35] J. Serrin, Local behavior of solutions of quasi-linear equations, *Acta Math.* **111** (1964), 247–302.
- [36] T. Toro, Doubling and flatness: geometry of measures, *Notices Am. Math. Soc.* **44** (1997), no. 9, 1087–1094.
- [37] N.S. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic equations, *Comm. Pure Appl. Math.* **20** (1967), 721–747.
- [38] P. Zamboni, Local boundedness of solutions of quasilinear elliptic equations with coefficients in Morrey spaces, *Boll. Un. Mat. Ital. B (7)* **8** (1994), no. 4, 985–997.

국문초록

이 학위논문에서는 발산형 준선형 m -라플라스형태의 연산자를 다루며, 비선형항은 적절한 모리 공간에 속하는 데이터들로 제어되는 카라데오도리 함수이다. 먼저 우리는 Ladyzhenskaya와 Ural'tseva의 구조를 모리 공간으로 확장한 방정식의 약해가 대역적 유계와 휠더 연속임을 보인다. 여기서 주어진 영역의 경계는 m -capacity 농도 조건을 만족한다. 또한, 우리는 (δ, R) -vanishing 성질을 만족하는 비선형항이 주어진 준선형 방정식의 약해에 대한 모리 공간상에서의 대역적 그래디언트 가늠을 이끌어낸다. 이 경우, 주어진 영역은 매끄럽지 않은 프랙탈 구조를 포함하는 라이펜버그 영역이다.

주요어휘: 준선형 타원형 방정식, 대역적 휠더 연속, 대역적 그래디언트 가늠, 모리 공간, 매끄럽지 않은 영역.

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